ANALYTICAL GEOMETRY:

WITH THE

PROPERTIES OF CONIC SECTIONS,

AND

AN APPENDIX,

CONSTITUTING A TRACT ON

DESCRIPTIVE GEOMETRY.

For the Use of the Royal Military College.

BY

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ADVERTISEMENT.

A Treatise of Analytical Geometry, with the Properties of the Conic Sections, being required for immediate use in this Institution, it has been found convenient so far to depart from the original arrangement as to publish a work under that title alone. The following Treatise must, therefore, be considered as the fourth of the series which is to constitute a General Course of Mathematics for the use of the gentlemen cadets and the officers in the senior department; and the Course, when completed, will now comprehend the subjects whose titles are subjoined: — I. Arithmetic and Algebra.* II. Geometry.* III. Plane Trigonometry with Mensuration.* IV. Analytical Geometry with the Properties of Conic Sections.* V. The Differential and Integral Calculus. VI. Practical Astronomy and Geodesy, including Spherical Trigonometry.* VII. The Principles of Mechanics; and VIII. Physical Astronomy.

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PREFACE.

THE properties of straight lines, as well as of plane figures bounded by straight lines or circles, and of solids bounded by planes, or by surfaces produced by the revolutions of straight lines or circles, have, in a part of the course already published for this Institution (Elements of Geometry), been demonstrated by the synthetical method of the ancient geometers; but in the present work, which contains investigations relating to the properties of curve lines and surfaces of more complex kinds, processes of an analytical nature have been employed. For such subjects the analytical method has great advantages over the other, both from the simplicity with which the steps occurring in the researches may be represented, and from the comprehensiveness of the results obtained; this last being such that the solutions of the different cases of a proposition are usually included in one general formula.

Descartes was the first who applied the processes of algebra to geometrical propositions: he observed that lines, straight or curved, and the curved surfaces of solids, afford relations between the co-ordinates of points in those lines and surfaces; and he was led to express lines and surfaces by algebraic equations; those which involve two variable quantities being capable of representing the positions of an infinite number of points in a straight line, or a plane curve, while equations involving three variables are capable of representing the positions of an infinite number of points in a line or surface of any kind, and situated in any manner in space: he, also, first arranged curves in orders according to the degrees of the variables in the equation. These discoveries gave rise to the branch of science called analytical geometry, which, in its actual state, is of the highest value as a means of investigating the properties of curve lines and surfaces.

In the present work it has been attempted to render the investigations as easy as possible for students who, in mathematics, are familiar with the processes of elementary geometry and algebra only; and it will, perhaps, be thought that, in some cases, this has been attended with a sacrifice of elegance. It is hoped, however, that the reason here given

will be considered as a sufficient excuse for that which otherwise might appear to be a defect.

The work consists of three sections, the first of which, designated Analytical Geometry of Two Dimensions, contains explanations of the manner of forming equations for straight lines, with the more useful propositions concerning such lines, and the transformation of co-ordinates in one plane. In the second, a general equation of the second degree with two variables being assumed, the positions of the co-ordinate axes corresponding to its different transformations are shown; the particular equations of the ellipse, the hyperbola, the parabola, and the circle are found; and the most useful properties of those curves are investigated in a series of propositions. The third section, entitled Analytical Geometry of Three Dimensions, contains the equations of straight lines and planes in space, with expressions for the inclinations of lines and planes to one another: the relations between plane figures and their orthogonal projections are determined, and formulæ for the transformation of co-ordinates in space are investigated. It contains, also, the equations for surfaces of revolution, and for curve surfaces which are not such, together with the equations for the curve lines formed by planes intersecting the surfaces.

The Appendix is a short tract on Descriptive Geometry, containing a series of the principal propositions relating to the intersections of straight lines and planes in space, the solutions of which are performed graphically by orthogonal projections of the lines, on two co-ordinate planes; and there are added the projections of the three most useful forms of stones for vaults of masonry.

The art of cutting stones into the forms required for the purposes of architecture and engineering is designated *Stereotomy;* and the representations, on two co-ordinate planes, of the projections of wrought stones, with the determinations of the figures of the sections made, when such stones are cut by planes passing through them in assigned directions, are among the many practical applications of descriptive geometry.

The author takes this opportunity of acknowledging his obligations to his friend and colleague, Professor Scott, who has liberally contributed the part which constitutes the first section of this Treatise.

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* In the figures 1 and 2. of this proposition, the line PQ has been, unintentionally, drawn through the point F.

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SECTION I.

ANALYTICAL GEOMETRY OF TWO DIMENSIONS.

CHAPTER I.

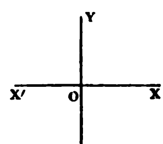
POSITION AND CO-ORDINATES OF A POINT IN A PLANE.—DISTANCE BETWEEN TWO POINTS.

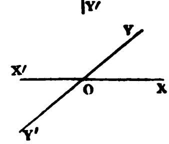
ARTICLE 1. Analytical Geometry is the name given to that branch of mathematics in which Algebra is made the instrument of geometrical investigation. It is divided into two parts — Analytical Geometry of Two Dimensions, and Analytical Geometry of Three Dimensions, according as the objects of investigation are situated in the same plane, or in any manner in space.

A mathematical question which admits of a limited number of solutions only, is said to be determinate; and indeterminate, if the number of solutions is unlimited.

Determinate problems have relation to the magnitude and position of geometrical quantities; indeterminate, to their figure.

In analytical geometry of two dimensions, the points of a line or of the contour of a figure are usually referred to two given straight lines, as $X'OX$, YOY', which intersect in a point O. $X'X$, YY' are termed axes. According as XOY is a right angle, or not a right angle, the axes are termed rectangular, or oblique.

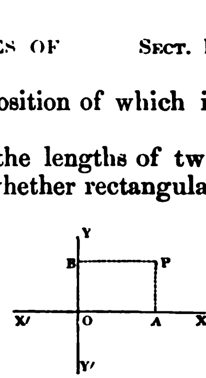


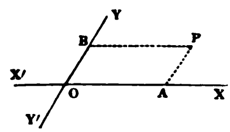


Sometimes also the points of a line or of the contour of a figure are referred to their distances from a given point, and the angles which the straight lines on which these distances are

measured make with a straight line the position of which is given.

2. The position of a point is given if the lengths of two lines drawn from it parallel to two axes, whether rectangular or oblique, are given.



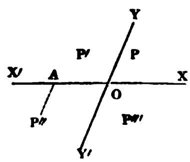


Let OX, OY be two axes at right angles to one another in the first figure, and oblique to one another in the second, and let the point P be in the same plane as the axes: then, if the lengths of the lines PA, PB drawn from P parallel to OY, OX respectively, be given, the position of P is given. For, by construction, the figure OP is a parallelogram whose sides are all given, OA being equal to PB, and OB to PA; and it is evident that, except P, no point (within the angle XOY) can be at once at the distance PA from the axis OX, and at the distance PB from the axis OY.

3. The segment OA which is cut off from the axis OX is called the abscissa; and PA which is drawn parallel to the axis OY, the ordinate of the point P. When mentioned together, and not distinguished from each other, OA, AP are called the co-ordinates of the point P.

The axis OX, produced if necessary towards X′, is called the axis of the abscissæ; the axis OY, produced if necessary towards Y′, the axis of the ordinates; and O, the point in which the axes intersect each other, the origin of the co-ordinates.

If the position of a point P is either unknown or variable, the abscissa is denoted by x, and the ordinate by y. The co-ordinates of a given point are denoted by those letters which are usually employed to represent known quantities, or by the letters x and y, accented (as x', y'; x'', y'') in such a manner as to be distinguishable from x and y when representing unknown or variable co-ordinates. The symbol (x, y) is employed to denote that the abscissa and ordinate of a point are lines of unknown or indefinite length; the symbol (a, b) to denote that the abscissa of a point is a given line a, and the ordinate another given line b.



If a point is situated in the axis of x, then $y = 0$: if in the axis of y, $x = 0$: if it coincides with the origin, $x = 0$, $y = 0$.

4. It is explained in Article 3. of the Treatise on Plane Trigonometry and Mensuration, that if the

values of lines measured along or parallel to OX, OY are supposed to be positive, the values of lines measured along or parallel to OX′, OY′ must be considered negative.

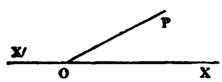
Thus, if the co-ordinates of the point P, in the angle XOY, are (x, y),
the co-ordinates of the point P′, in the angle X′OY, are $(-x, y)$,
the co-ordinates of the point P″, in the angle X′OY′, are $(-x, -y)$,
and the co-ordinates of the point P‴, in the angle XOY′ are $(x, -y)$.

The signs of the co-ordinates of a point indicate in which of the four angles formed by the intersection of the axes that point is situated.

Employing these conventions, the expression $(0, b)$ denotes a point in the axis OY, at a distance b, from the origin: $(-c, 0)$, a point in the negative axis of x, at a distance c, from the origin: $(-x', -y')$, a point in the angle X′OY′, at a distance x', from the axis of y, and a distance y', from the axis of x. To determine the position of the point $(-x', -y')$, for example, it is necessary to take along the axis OX′, a distance OA, $= x'$, and from A to draw AP″ parallel to OY′ and equal to y': P″ is the point required. If the numerical value of x' is 5, and that of y', 2, it will be requisite to make OA $= 5$, and AP″ $= 2$ times the length of the unit of linear measure.

5. The position of a point is also determined, if its distance from a given point and the angle which a straight line joining the given point and the point which is to be determined, makes with a straight line given in position, are given.

Let the straight line X′X, and the point O, in X′X, be given in position; then, if an angle equal to the angle XOP, and the length of the straight line OP which joins the given point O and the required point P, are given, the position of the point P is given. For making the angle XOP equal to the given angle, and the length of OP equal to that of the given line, it is evident that the point P is determined.



The point O is called the pole; the line OX, the initial line; the distance OP, the radius vector of the point P; and the angle XOP, the angle of revolution; also, the line OP and the angle XOP are called the polar co-ordinates of the point P.

The radius vector and angle of revolution, when indeterminate quantities, are represented by r and θ respectively.

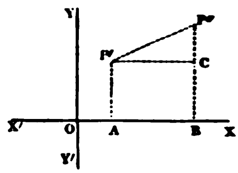
Admitting that r may be of any length from 0 to ∞, and that θ may have any magnitude from 0 to $+360°$ or from 0 to $-360°$, it is evident that the polar co-ordinates (r, θ) are

capable of indicating any point in a plane passing through the initial line X′X.

6. To express the distance between two points in terms of the co-ordinates of these points.

Let P′, P″ be two given points whose co-ordinates are (x', y'), (x'', y'') respectively: P′, P″ being joined, it is required to express the distance P′P″ in terms of x', y', x'', y''.

1st. Let the axes OX, OY be rectangular. From P′, P″ draw P′A, P″B, perpendicular to OX; and from P′ draw P′C, parallel to OX:



Then, since P′CP″ is a right angle,

$$P'P''^2 = P''C^2 + P'C^2. \text{ (Euc. I. 47.)}$$

$$\therefore P'P'' = \sqrt{P''C^2 + P'C^2}.$$

Now $P'C = AB = OB - OA = x'' - x'$,

and $P''C = P''B - CB = P''B - P'A = y'' - y'$;

substituting these expressions for P′C, P″C, and denoting P′P″ by D,

$$D = \sqrt{\{(x''-x')^2 + (y''-y')^2\}}$$

The ambiguous sign ($\pm$) is not prefixed to the radical, because the absolute distance only, between the two points, is required.

If the point P′ is supposed to be situated in the axis OX, at a distance x' from the origin, the co-ordinates of P′ are $(x', 0)$, and

$$D = \sqrt{\{(x''-x')^2 + (y''-0)^2\}} = \sqrt{\{(x''-x')^2 + y''^2\}}.$$

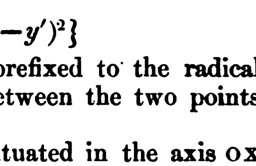
If in the axis OY, at a distance y' from the origin, the co-ordinates of P′ are $(y', 0)$; and

$$D = \sqrt{\{(x''-0)^2 + (y''-y')^2\}} = \sqrt{\{x''^2 + (y''-y')^2\}}.$$

If in the origin, the co-ordinates of P′ are (0, 0) and

$$D = \sqrt{\{x''^2 + y''^2\}}$$

Let P′ be situated in the angle YOX′, at distances x', y' from the axis OY, OX′ respectively. In this case, the co-ordinates of P′ are $(-x', y')$.



The figure being constructed as before,

$P'C = AB = OB + OA \quad = x'' + x'$,

$P''C = P''B - CB = P''B - P'A = y'' - y'$;

$P'P''$, or D, $= \sqrt{\{(x''+x')^2 + (y''-y')^2\}}$.

If $(-x', y')$ be substituted for (x', y') in the expression $D = \sqrt{\{(x''-x')^2 + (y''-y')^2\}}$, the result is

$$D = \sqrt{\{[x''-(-x')]^2 + (y''-y')^2\}},$$

$$\text{or} \quad D = \sqrt{\{(x''+x')^2 + (y''-y')^2\}}.$$

From this result it follows that the signs of direction are subject to the ordinary rules of Algebra, when the quantities which are affected by them are connected by the signs which are employed in Algebra, and these signs are used in the algebraic sense.

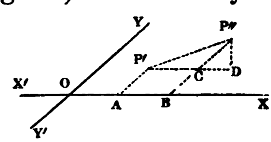
a. The angle which the line joining the points P′, P″, makes with the axis XX′ may be expressed in terms of (x', y'), (x'', y''), the co-ordinates of P′, P″. For, since P′C is parallel to X′X, the angle P″P′C is equal to the angle which P′P″ makes with X′X (Euc. I. 29.). Now $\frac{\text{P}''\text{C}}{\text{P}'\text{C}} = \tan.\ \text{P}''\text{P}'\text{C}$ (Trigon., Art. 17.); or, replacing P″C, P′C, by their values, and putting θ for the angle P″P′C, we have

$$\frac{y''-y'}{x''-x'} = \tan.\ \theta.$$

This formula is applicable to the line P′P″ in both figures, x' being considered as negative for the position of P′ in the second figure.

7. 2nd. Let the axes X′X, YY′ be oblique, also let the angle XOY (which is supposed to be given) be denoted by ω.

Draw P′A, P″B parallel to OY, and P′C parallel to OX; also draw P″D perpendicular to OX, meeting P′C produced in D. Then, as in the case of rectangular axes,



$$\text{P}'\text{C} = x''-x',\ \text{P}''\text{C} = y''-y'.$$

and (Euc. I. 29.) the angle P″CD = XOY:

but (Euc. II. 12.) $\text{P}'\text{P}''^2 = \text{P}'\text{C}^2 + \text{P}''\text{C}^2 + 2\,\text{P}'\text{C}.\ \text{CD}$;

whence (Trigon., Art. 56.) $\text{P}'\text{P}''^2 = \text{P}'\text{C}^2 + \text{P}''\text{C}^2 + 2\,\text{P}'\text{C}\ .\ \text{P}''\text{C}\ \cos.\ \text{P}''\text{CD}$,

or $\quad \text{D}^2 = (x''-x')^2 + (y''-y')^2 + 2\,(x''-x')\,(y''-y')\cos.\ \omega$;

therefore $\quad \text{D} = \sqrt{\{(x''-x')^2 + (y''-y')^2 + 2\,(x''-x')\,(y''-y')\cos.\ \omega\}}$.

If the co-ordinates of P′ are $(-x', y')$,

$$\text{D} = \sqrt{\{(x''+x')^2 + (y''-y')^2 + 2(x''+x')\,(y''-y')\cos.\ \omega\}};$$

and if the co-ordinates of P′ are (O, O),

$$\text{D} = \sqrt{(x''^2 + y''^2 + 2\,x''y''\cos.\ \omega)}.$$

a. The angle $(=\text{P}''\text{P}'\text{C})$ which the line joining the points P′, P″ makes with the axis XX′ may be expressed in terms of (x', y'), (x'', y'') the co-ordinates of those points and of ω, the inclination of the axes. For in the triangle P″P′C we have (Trigon. Art. 57.)

$$\frac{\sin.\ \text{P}''\text{P}'\text{C}}{\sin.\ \text{P}'\text{P}''\text{C}} = \frac{\text{P}''\text{C}}{\text{P}'\text{C}},$$

in which, putting θ for the angle P″P′C, the angle P′P″C $=(\omega-\theta)$;

that is, $\dfrac{\sin.\ \theta}{\sin.(\omega-\theta)}=\dfrac{y''-y'}{x''-x'}$; or, developing sin. $(\omega-\theta)$. (Trigon. Art. 31.)

$$\frac{\sin.\ \theta.}{\sin.\ \omega\cos.\ \theta-\cos.\ \omega\sin.\ \theta}=\frac{y''-y'}{x''-x'}:$$

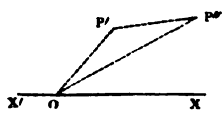
or, again, dividing every term in the first member by cos. θ, and substituting tan. θ. for $\dfrac{\sin.\ \theta}{\cos.\ \theta}$ (Trigon. Art. 19. *c.*)

$$\frac{\tan.\ \theta}{\sin.\ \omega-\tan.\ \theta\cos.\ \omega}=\frac{y''-y'}{x''-x'};$$

whence, after reduction,

$$\tan.\ \theta=\frac{(y''-y')\sin.\ \omega}{(x''-x')+(y''-y')\cos.\ \omega}=\frac{\dfrac{y''-y'}{x''-x'}\sin.\ \omega}{1+\dfrac{y''-y'}{x''-x'}\cos.\ \omega}.$$

When ω is a right angle, in which case the co-ordinate axes are rectangular, either of these expressions for the value of tan. θ becomes identical with that which was found above. (Art. 6. *a.*)



8. 3rd. Let the co-ordinates be polar, and let (r', θ'), (r'', θ'') be the co-ordinates of P′, P″ respectively.

Join P′, P″; then OP′ $=r'$, OP″ $=r''$, and the angle P′OP″ (= XOP′ − XOP″) $=\theta'-\theta''$.

But P′P″ $=\sqrt{(\text{OP}'^2+\text{OP}''^2-2\ \text{OP}'.\ \text{OP}''\cos.\ \text{P}'\text{OP}'')}$; therefore, by substitution,

$$\text{D}=\sqrt{(r'^2+r''^2-2\ r'\ r''\cos.\ (\theta'-\theta''))}.$$

In analytical investigations, rectangular axes commonly lead to the most simple results; for which reason they are generally employed. If oblique axes are used, the circumstance is always mentioned; and in this case, the inclination of the axes is either given, or left to be assumed at pleasure.

When polar co-ordinates are employed, the initial position of the radius vector is either given, or left to be assumed at pleasure.

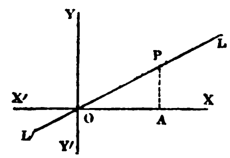
CHAP. II.

OF THE EQUATION OF A STRAIGHT LINE IN A PLANE.

9. An equation which expresses, in a general manner, the constant relation between the ordinate and abscissa of a line, is termed the Equation of that line. Conversely, the line is termed the Locus of the equation.

Let LL′ be a straight line; and first, let LL′ pass through O, the origin of the co-ordinates on the axes X′X, YY′. In LL′ take any point P, and draw PA parallel to OY.

Fig. 1.



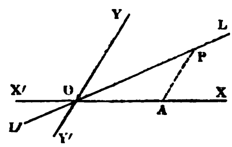


Fig. 2.

Then, if from any other points taken in LL′, straight lines, parallel to the axis of y, are drawn to meet the axis of x, the parallels to the axis of y are the ordinates, and the portions of the axis of x, intercepted between O and the feet of the ordinates, are the corresponding abscissæ of these points.

But the ratio of each ordinate to its abscissa is expressed by the ratio $\frac{PA}{OA}$. (Euc. VI. 4.) Wherefore the ratio $\frac{PA}{OA}$ expresses a constant relation between the ordinate and abscissa of every point in the straight line LL′.

If the axes are rectangular (Fig. 1.) $\frac{PA}{OA}=\tan. XOP$, (Trigon. Art. 17.)

and $PA=OA. \tan. XOP$. 1

If the axes are oblique (Fig. 2.) $\frac{PA}{OA}=\frac{\sin. XOP}{\sin. OPA}$, (Trigon. Art. 57.)

and $PA=\frac{\sin. XOP}{\sin. OPA}. OA$ 2

Denoting OA by x, AP by y; the angles XOY, XOL by ω, θ, respectively; and therefore APO $(=YOP=XOY-XOL)$, by $\omega-\theta$, formulæ 1, 2, become

$y = \tan. \theta \times x$ 3

$y = \dfrac{\sin. \theta}{\sin. (\alpha - \theta)} x$ 4

By giving the angle θ every value from 0 to $+360°$, or from 0 to $-360°$, the formulæ 3 and 4 may each be made to express the relation of the co-ordinates of any straight line which is in the plane of the axes X'X, YY', and passes through the point O.

The coefficient of x in formula 3, in which the axes are rectangular, is different from the coefficient of x in formula 4, in which the axes are oblique; but both coefficients are generally denoted by the letter α; and the two equations are thus comprehended under the more general formula, $y = \alpha\, x$.

Since (Arts. 6. *a.* and 7. *a.*) $\dfrac{y'' - y'}{x'' - x'}$, is the equivalent of $\tan. \theta$, or of $\dfrac{\sin. \theta}{\sin. (\omega - \theta)}$ according as the co-ordinate axes are rectangular or oblique, and that either of these trigonometrical terms is denoted by α; we have, on substituting α for $\dfrac{y'' - y'}{x'' - x'}$ in the last value of $\tan. \theta$ (Art. 7. *a.*), when the axes are oblique,

$$\tan. \theta = \frac{\alpha \sin. \omega}{1 + \alpha \cos. \omega}.$$

10. 2nd. Let the straight line LL' cut the axis of y in any point B: through B draw BD parallel to the axis of x. In LL' take any point P; draw PC parallel to OY, and let PC cut BD in E. Since BD is parallel to OX, and PC to OY; BE $=$ OC, EC $=$ OB; the angle EBP $= \theta$, and BPE $=$ PBY $= \omega - \theta$.

Fig. 1.

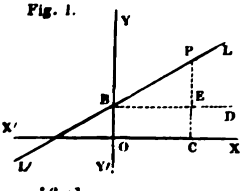
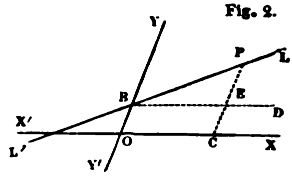


Fig. 2.



Then, if the axes are rectangular (Fig. 1.),

PE $=$ BE tan. PBE $=$ OC tan. PBE,

and PC $=$ OC tan. PBE $+$ EC $=$ OC tan. PBE $+$ OB,

or $y = x \tan. \theta + \text{OB}$ 5

If the axes are oblique (Fig. 2.),

$$\text{PE} = \frac{\sin. \text{PBE}}{\sin. \text{BPE}} \times \text{BE} = \frac{\sin. \text{PBE}}{\sin. (\text{DBY} - \text{DBL})} \times \text{OC},$$

and $PC = \frac{\sin. PBE}{\sin. (DBY - DBL)} \times OC + EC = \frac{\sin. PBE}{\sin. (DBY - DBL)} \times OC + OB$,

$$\text{or, } y = \frac{\sin. \theta}{\sin. (\omega - \theta)} \times x + OB \quad . \quad . \quad . \quad . \quad 6$$

The line OB (which is called the ordinate of the origin) is usually denoted by b. Representing the coefficient of x, as in equations 3 and 4, by α, equations 5 and 6 are reduced to the general expression,

$$y = \alpha\, x + b.$$

If LL′ cuts the axis of y below the origin, the sign of OB, or b, will be negative.

11. If to b are given all values from 0 to $+\infty$, and from 0 to $-\infty$, and if to the angle θ are likewise given (as in the 1st case) all values from 0 to $+360°$ or from 0 to $-360°$, it is manifest that the formula $y = \alpha\, x + b$ (or $y = \pm \alpha\, x \pm b$) is rendered sufficiently general to express every relation which can subsist between the ordinates and abscissæ of all straight lines in the plane of the axes X′X, Y′Y.

If α is constant and b variable, $y = \alpha\, x + b$ is the equation of all straight lines parallel to that straight line which passes through the origin, and makes an angle θ with the axis of x.

If b is constant and α variable, $y = \alpha\, x + b$ is the equation of any straight line which cuts the axis of y at a distance b from the origin, and which may, consequently, make a complete revolution round that point (as a centre) in which it cuts the axis of y.

If $\alpha = 0$, $y = b$, a result which shows that when $\alpha = 0$, no relation can be established between x and y. But when $\alpha = 0$;—

If the axes are rectangular, $\tan. \theta = 0$; consequently the angle $\theta = 0$. Whence LL′ makes an angle $= 0$ with the axis of x; that is, LL′ does not cut the axis of x, and is therefore parallel to it. Also:

If the axes are oblique, and $\alpha = 0$, $\frac{\sin. \theta}{\sin. (\omega - \theta)} = 0 \therefore \sin. \theta = 0$,

and the angle $\theta = 0$. Whence LL′ makes an angle $= 0$ with the axis of x, or is parallel to the axis of x.

Consequently, whether the axes are rectangular or oblique, $y = 0 \times x + b$ is the equation of a straight line which cuts the axis of y at a distance b from the origin, and is parallel to the axis of x.

If $\alpha = \infty$, the result $y = \infty \times x + b$, or $\frac{y - b}{\infty} = x$, or $0 = x$, shows that also, when $\alpha = \infty$, no relation can be established between x and y.

But when $\alpha = \infty$; —

If the axes are rectangular, tan. $\theta = \infty =$ tan. $90°$. Whence LL′ is perpendicular to the axis of x, or parallel to the axis of y. And:

If the axes are oblique, $\frac{\sin.\theta}{\sin.(\omega-\theta)} = \infty$; wherefore $\sin.(\omega-\theta)=0$

$$\text{or } \sin.\omega \cos.\theta - \cos.\omega \sin.\theta = 0$$
$$\therefore \sin.\omega \cos.\theta = \cos.\omega \sin.\theta$$
$$\text{and } \frac{\sin.\omega}{\cos.\omega} = \frac{\sin.\theta}{\cos.\theta} \text{ or } \tan.\omega = \tan.\theta.$$

Whence the angles ω, θ, are equal to one another: and LL′ is parallel to the axis of y. Consequently, when $\alpha = \infty$, $y = \infty x + b$ is the equation of a line which is parallel to the axis of y.

If $b=0$; the line LL′ passes through the origin: and if $b=\infty$, the line LL′ cuts the axis of y in a point, which is at an infinite distance from the origin.

From the preceding observations it is evident that the values of x and y are different for every point in the same straight line: that the values of α and b are constant for every point in the same straight line, but different for different straight lines which are referred to the same axes: that the value of α is the same, but that of b different, for all straight lines which make the same angle with the axis of x: and that the value of α is different, but that of b the same, for all the straight lines which pass through the same point in the axis of y.

By the different values, therefore, of α and b are different straight lines, which are referred to the same axes distinguished from each other.

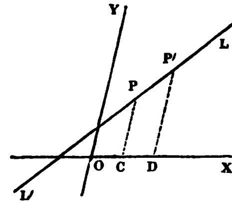
In the remaining part of this chapter, the expression $y = \alpha x + b$ (in which α, b, may be positive, negative, equal to zero, or equal to infinity) is employed to represent the equation of a straight line generally; and $y = \alpha' x + b'$, $y = \alpha'' x + b''$, &c., the equations of particular lines.

12. To trace a straight line from its equation.

Let $y = \alpha' x + b'$ be the equation of a straight line. The values of α', b' (which are given) are numbers. If in the indeterminate equation $y = \alpha' x + b'$ two arbitrary values, x', x'', are given to x, the results y', y'' are two corresponding determinate values of y (Alg. Art. 92.).

(x', y'), (x'', y''), are therefore two points in the line. The positions of these points being fixed (Art. 2.), the straight line traced through them is that which is required.

Example. Given $y=\frac{1}{2}x+2\frac{1}{2}$ to trace the straight line.
Let $x'=1$ $\therefore$ $y'=\frac{1}{2}\times 1+2\frac{1}{2}=\frac{1}{2}+2\frac{1}{2}=3$,
Let $x''=2$ $\therefore$ $y''=\frac{1}{2}\times 2+2\frac{1}{2}=1+2\frac{1}{2}=3\frac{1}{2}$.



Let OX, OY be the axes: from OX cut off OC$=1$, and through C draw CP parallel to OY, making it$=3$. P is the point (x', y'). Again, from OX cut off OD$=2$; and through D draw DP′ parallel to OY, making it$=3\frac{1}{2}$. P′ is the point (x'', y''). Through PP′ draw the straight line LP′PL′. LP′PL′ is the straight line whose equation is $y=\frac{1}{2}x+2\frac{1}{2}$.

A straight line may be also traced by finding the points in which it cuts the axes, and drawing it through these points.

These points are found by making successively $x=0$, $y=0$ in the equation of the line; thus: If in the equation $y=a'x+b'$, $x=0$; $y=b'$

$$\text{and if } y=0; \quad x=-\frac{a'}{b'}$$

$(0, b')$ and $(-\frac{a'}{b'}, 0)$ are consequently the co-ordinates of the points in which the straight line $y=a'x+b'$ cuts the axes; and the straight line traced through these points is that which is required.

When the equation is $y=a'x$, if $x=0$; $y=0$. Making, in this equation, $x=1$; $y=a'$. Whence $(0, 0)$, $(1, a')$ are the co-ordinates of two points in the line.

13. The indeterminate equation of the first degree involving two variable quantities x, y, is $Ax+By+C=0$, in which A, B, C, are constants. The locus of this equation is a straight line.

Transposing the terms of this equation, $By=-Ax-C$.

Dividing by B . . . $y=\frac{-A}{B}x-\frac{C}{B}$.

Making $-\frac{A}{B}=a$, and $-\frac{C}{B}=b$, this equation is reduced to

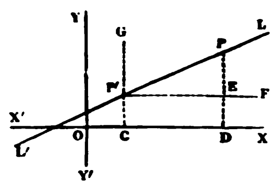
$$y=ax+b.$$

Hence it appears that the equation $Ax+By+C=0$, may in all cases be made to coincide with that of a straight line. But an equation is said to represent a locus when it is of the form, or is reducible to the form, of the equation to that locus: whence the locus of the equation $Ax+By+C=0$, is a straight line.

PROBLEMS RELATING TO THE STRAIGHT LINE.

14. PROB. 1. To find the equation of a straight line which passes through a given point (x', y') and makes a given angle, θ, with the axis of x.

Let LL′ be the line, and P′ the point, the co-ordinates of which, referred to the axes OX, OY, are (x', y').



Through P′ draw P′F parallel to OX and produce CP′, the ordinate of the given point, to G.

Let P be any other point in LL′. From P draw the ordinate PD, and let OD, PD, the co-ordinates of the indeterminate point P, be denoted by x, y, respectively.

Then the angle EP′G $=\omega$; EP′L $=\theta$; LP′G $=$ P′PE $=\omega-\theta$; also PE $=$ PD $-$ DE $=$ PD $-$ P′C $= y - y'$;

$$\text{P}'\text{E} = \text{CD} = \text{OD} - \text{OC} = x - x'.$$

But PE $=$ tan. PP′E $\times$ P′E $=$ tan. $\theta \times$ P′E $= \alpha' \times$ P′E, if the axes are rectangular;

or $\text{PE} = \dfrac{\sin.\ \text{PP}'\text{E}}{\sin.\ \text{P}'\text{PE}} \times \text{P}'\text{E} = \dfrac{\sin.\ \theta}{\sin.(\omega-\theta)} \times \text{P}'\text{E} = \alpha' \times \text{P}'\text{E}$, if the axes are oblique.

Whence substituting for PE, P′E, their values,

$$y - y' = \alpha' (x - x').$$

a. If the angle θ is supposed variable, α' becomes α; whence $y - y' = \alpha (x - x')$ is the equation of every straight line which can be drawn through the given point (x', y'). If the co-ordinates of the given point are $(0, b')$, the equation becomes

$$y - b' = \alpha' (x - 0), \text{ or } y = \alpha' x + b'.$$

b. Since the equation of every straight line which passes through the given point (x', y') is $y - y' = \alpha (x - x')$; and, the co-ordinate axes being rectangular, the tangent of the angle which the straight line passing through the two given points (x', y'), (x'', y'') makes with the axis of x is $\dfrac{y'' - y'}{x'' - x'} = \alpha$ (Art. 6. *a.*); it follows that the equation of the straight line which passes through two given points is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x').$$

The same expression denotes the equation of a straight line passing through two given points (x', y'), (x'', y'') when the co-ordinate axes are not at right angles to one another; but, in this case, the coefficient $\frac{y''-y'}{x''-x'}$ of $(x-x')$ is equivalent to $\frac{\sin.\theta}{\sin.(\omega-\theta)}$.

15. PROB. 2. To find the equation of a straight line which passes through a given point (x', y') and is parallel to a given straight line $y''=\alpha' x''+b'$.

Since the required line passes through the point (x', y'), its equation is of the form $y-y'=\alpha(x-x')$. (Art. 14. a.)

Also since it is parallel to the given line $y''=\alpha' x''+b'$, the angle which it makes with the axis of x is equal to the angle which the given line makes with the axis of x. (Euc. I. 29.) Whence $\alpha=\alpha'$

$$\text{and } y-y'=\alpha'(x-x'),$$

is the equation of the straight line which passes through the point (x', y'), and is parallel to the given straight line $y''=\alpha' x''+b'$.

16. PROB. 3. Let $y=\alpha' x+b'$, and $y=\alpha'' x+b''$ be the equations of two straight lines which cut each other: it is required to find the co-ordinates of the point of intersection of these lines, and the angle which they make with each other.

Since the point in which the straight lines cut each other is common to both the lines, the values of the co-ordinates of that point are the same in the equations of both the lines. Whence, for the point of intersection, and for that point only, the y and the x of the equation $y=\alpha' x+b'$ are simultaneously equal to the y and the x respectively of the equation $y=\alpha'' x+b''$.

To obtain the expressions of these particular values of x, y, in terms of α', b', α'', b'', the given quantities in the equations of the two lines,

$$\text{from } y=\alpha' x+b' \text{ subtract } y=\alpha'' x+b''$$
$$\therefore (\alpha''-\alpha')\,x=b'-b''.$$

Whence $x=\frac{b'-b''}{\alpha''-\alpha'}$ is the abscissa of the point of intersection.

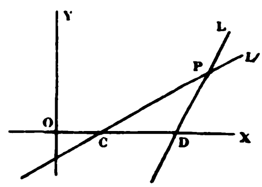
Substituting $\frac{b'-b''}{a''-a'}$ for x, in the equation of the first line,

$$y=a'\frac{b'-b''}{a''-a'}+b'=\frac{a''b'-a'b''}{a''-a'}.$$

Wherefore the co-ordinates of the point of intersection are

$$\left(\frac{b'-b''}{a''-a'}, \frac{a''b'-a'b''}{a''-a'}\right).$$

Next, to find the angle (ϕ) which the straight lines make with each other.



If the axes are rectangular,

a' = tan. PDX or tan. θ' (Art. 9.)

a'' = tan. PCX or tan. θ''

But CPD = PDX − PCX. (Euc. I. 32.)

or $\phi = \theta - \theta''$

$$\therefore \text{tan. } \phi = \text{tan. } (\theta'-\theta'') = \frac{\text{tan. } \theta' - \text{tan. } \theta''}{1+\text{tan. } \theta' \times \text{tan. } \theta''}, \text{(Trigon. Art. 37.)}$$

$$\text{or tan. } \phi = \frac{a'-a''}{1+a'a''}. \qquad (a)$$

If the straight lines are parallel, $\phi = 0 \therefore \text{tan. } \phi = 0 \therefore a' = a''$, as in Art. 15.

If $\phi = 90°$, tan. $\phi = \infty \therefore 1 + a'a'' = 0$, and $a'' = -\frac{1}{a'}$.

This is the condition which must be satisfied, in order that two straight lines may be perpendicular to each other. If the first line is perpendicular to the axis of x, $a' = \infty$, and $a'' = -\frac{1}{\infty} = 0$; a result from which it follows that the second line is parallel to the axis of x.

If the axes are oblique, $\text{tan. } \theta' = \frac{a' \text{ sin. } \omega}{1+a' \text{ cos. } \omega}$ (Art. 9.)

$$\text{tan. } \theta'' = \frac{a'' \text{ sin. } \omega}{1+a'' \text{ cos. } \omega}.$$

Whence

$$\text{tan. } (\theta'-\theta'') \text{ or tan. } \phi = \frac{\frac{a' \text{ sin. } \omega}{1+a' \text{ cos. } \omega} - \frac{a'' \text{ sin. } \omega}{1+a'' \text{ cos. } \omega}}{1+\frac{a' \text{ sin. } \omega}{1+a' \text{ cos. } \omega} \times \frac{a'' \text{ sin. } \omega}{1+a'' \text{ cos. } \omega}},$$

$$\text{or tan. } \phi = \frac{(a'-a'') \text{ sin. } \omega}{1+a'a''(\text{cos.}^2 \omega + \text{sin.}^2 \omega) + (a'+a'') \text{ cos. } \omega},$$

$$\text{or tan. } \varphi = \frac{(\alpha' - \alpha'') \sin. \omega}{1 + \alpha'\alpha'' + (\alpha' + \alpha'') \cos. \omega}. \qquad (b)$$

If the two straight lines are at right angles to each other: tan. $\varphi =$ tan. $90° = \infty$: whence the denominator of the value of tan. φ is equivalent to 0;

$$\text{that is, } 1 + \alpha'\alpha'' + (\alpha' + \alpha'') \cos. \omega = 0$$

$$\therefore \alpha'' = -\frac{1 + \alpha' \cos. \omega}{\alpha' + \cos. \omega}.$$

If, also, the axes are rectangular, $\omega = 90°$ and cos. $\omega = 0$.

Whence, in this case $\alpha'' = -\frac{1}{\alpha'}$, as before.

a. It follows, the co-ordinate axes being rectangular, that if the equation of a straight line is $y = \alpha' x + b'$, that of a straight line at right angles to it will be

$$y = -\frac{1}{\alpha'} x + b''.$$

17. PROB. 4. To find the equation of a straight line which passes through the given point (x', y') and makes a given angle, ϕ, with the given straight line $y = \alpha' x + b'$.

Since the required line passes through the point (x', y') its equation is of the form $y - y' = \alpha (x - x')$ (Prob. 1. *a.*).

The only quantity, in this equation, which it is necessary to determine is α.

If the axes are rectangular, α denotes the trigonometrical tangent of the angle which the required line makes with the axis of x.

If this line is L′C (see fig. of Prob. 3.), $\alpha =$ tan. L′CX $=$ tan. θ; LD being the given line, $\alpha' =$ tan. LDX $=$ tan. θ'.

Whence, also, CPD $= \varphi$.

But L′CX $=$ LDX $-$ CPD or $\theta = \theta' - \varphi$ (Euc. I. 32.)

$$\therefore \tan. \theta = \frac{\tan. \theta' - \tan. \varphi}{1 + \tan. \theta' \times \tan. \varphi}. \quad \text{(Trigon. Art. 37.)}$$

or, denoting tan. φ by t' and writing α, α' for tan. θ, tan. θ', respectively,

$$\alpha = \frac{\alpha' - t'}{1 + \alpha' t'}.$$

Substituting this value of α in the equation $y - y' = \alpha (x - x')$

$$y - y' = \frac{\alpha' - t'}{1 + \alpha' t'} (x - x').$$

This is the equation of a straight line which passes through the given point (x', y'), and makes a given angle φ, the tangent of which is t', with the given straight line $y = \alpha' x + b'$: the axes being rectangular.

If L′C is perpendicular to LD, $\varphi = 90°$, and tan. φ or $t' = \infty$. Whence

$$\frac{\alpha' - t'}{1 + \alpha' t'} = \frac{\frac{\alpha'}{\infty} - \frac{\infty}{\infty}}{\frac{1}{\infty} + \frac{\alpha' \times \infty}{\infty}} = -\frac{1}{\alpha'}, \text{ and } y - y' = \frac{\alpha' - t'}{1 + \alpha' t'}(x - x')$$

is reduced to $y - y' = -\frac{1}{\alpha'}(x - x')$. (*a*)

This is the equation of a straight line which passes through a given point (x', y') and makes a right angle with a straight line whose equation is $y = \alpha' x + b'$.

If the co-ordinate axes are oblique, α or α' denotes (Art. 14.) the ratio between the sines of the angles which the required line makes with the axes; and if this line is L′C, we shall have (Art. 9.)

$$\tan.\ \theta\ (= \tan.\ \text{L'CX}) = \frac{\alpha \sin.\ \omega}{1 + \alpha \cos.\ \omega};$$

$$\text{also } \tan.\ \theta'\ (= \tan.\ \text{LDX}) = \frac{\alpha' \sin.\ \omega}{1 + \alpha' \cos.\ \omega}.$$

Therefore the angle CPD being represented by φ as before, also α and θ now holding the places of α'' and θ'' respectively in the equivalents of tan. φ (Art. 16.), we obtain from the value of tan. φ at (*b*) in that Article,

$$\tan.\ \varphi + \alpha' \alpha \tan.\ \varphi + (\alpha' + \alpha) \tan.\ \varphi \cos.\ \omega = (\alpha' - \alpha) \sin.\ \omega:$$

hence

$$\alpha \{\sin.\ \omega + (\alpha' + \cos.\ \omega) \tan.\ \varphi\} = \alpha' \sin.\ \omega - (1 + \alpha' \cos.\ \omega) \tan.\ \varphi$$

or

$$\alpha = \frac{\alpha' \sin.\ \omega - (1 + \alpha' \cos.\ \omega) \tan.\ \varphi.}{(\sin.\ \omega + (\alpha' + \cos.\ \omega) \tan.\ \varphi}$$

It follows that, when the co-ordinate axes are oblique, the equation for a straight line which passes through a point (x', y'), and makes a given angle φ with the straight line $y = \alpha' x + b'$ is

$$y - y' = \frac{\alpha' \sin.\ \omega - (1 + \alpha' \cos.\ \omega) \tan.\ \varphi}{\sin.\ \omega + (\alpha' + \cos.\ \omega) \tan.\ \varphi}(x - x').$$

If $\varphi = 90°$, tan. $\varphi = \infty$ and α becomes equal to $-\frac{1 + \alpha' \cos.\ \omega}{\alpha' + \cos.\ \omega}$, as above (Art. 16.): also

$$y - y' = -\frac{1 + \alpha' \cos.\ \omega}{\alpha' + \cos.\ \omega}(x - x'). \qquad (b)$$

This is the equation for a straight line which passes through a given point (x', y') and makes a right angle with the straight line $y=\alpha' x+b'$; the axes being oblique.

18. PROB. 5. Given the equation of a straight line, $y=\alpha' x+b'$, and a point (x', y') without it, to find the length of the perpendicular drawn from the given point to the given straight line.

The equation of the straight line which passes through the point (x', y') and makes right angles with the straight line $y=\alpha' x+b'$, is

$y-y'=-\frac{1}{\alpha'}(x-x')$, if the axes are rectangular, and

$y-y'=-\frac{1+\alpha' \cos.\omega}{\alpha'+\cos.\omega}(x-x')$, if the axes are oblique (Prob. 4.)

1st. Let the axes be rectangular.

Then x', y' being the co-ordinates of one extremity of the line whose length is required, if the values of $x_{,}$, $y_{,}$, the co-ordinates of the point of intersection of the two lines are determined, and the differences $x_{,}-x'$, $y_{,}-y'$ are substituted for $x'-x''$ and $y'-y''$ in the formula $D=\sqrt{\{(x'-x'')^2+(y'-y'')^2\}}$ (Art. 6.), the result will be that which is required.

Since $x_{,}$, $y_{,}$, the co-ordinates of the point of intersection, are common to both lines, the equations of the lines for that point are

$$y_{,}=\alpha' x_{,}+b' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1$$

$$y_{,}-y'=-\frac{1}{\alpha'}(x_{,}-x') \quad . \quad . \quad . \quad . \quad . \quad . \quad 2$$

Multiplying the terms of eq. 1. by α', the terms of eq. 2. by $-\alpha'$, and adding the results,

$$\alpha' y'=(\alpha'^2+1)x_{,}-x'+\alpha' b'$$

$$\therefore x_{,}=\frac{\alpha' y'+x'-\alpha' b'}{\alpha'^2+1}.$$

$$\text{And } x_{,}-x'=\frac{\alpha' y'+x'-\alpha' b'}{\alpha'^2+1}-x'=\frac{\alpha'(y'-\alpha' x'-b')}{\alpha'^2+1}.$$

Substituting this value of $x_{,}-x'$ in equation 2.

$$y_{,}-y'=-\frac{1}{\alpha'}\times\frac{\alpha'(y'-\alpha' x'-b')}{\alpha'^2+1}=-\frac{y'-\alpha' x'-b'}{\alpha'^2+1}.$$

Substituting these values of $x_{,}-x'$, $y_{,}-y'$ in the formula which expresses the distance between two points, in terms of the co-ordinates of these points, it is found that the length of

the perpendicular from the point (x', y') on the straight line $y = a'x + b'$, is

$$\text{D} = \pm\sqrt{\left\{\frac{a'^2(y'-a'x'-b')^2}{(a'^2+1)^2} + \frac{(y'-a'x'-b')^2}{(a'^2+1)^2}\right\}} = \pm\sqrt{\left\{\frac{(a'^2+1)(y'-a'x'-b')^2}{(a'^2+1)^2}\right\}}$$

$$= \pm\sqrt{\left\{\frac{(y'-a'x'-b')^2}{a'^2+1}\right\}} = \pm\frac{y'-a'x'-b'}{\sqrt{(a'^2+1)}}.$$

As the value of D is positive, it is necessary to take the upper sign when the numerator $y'-a'x'-b'$ is positive, and the lower when it is negative.

If the point (x', y') coincides with the origin, $x'=0$, $y'=0$, and

$$\text{D} = \pm\frac{b'}{\sqrt{(a'^2+1)}}.$$

2nd. Let the axes be oblique.

Putting the equation of the given line, $y_{,} = a'x_{,} + b'$, under the form

$$y_{,} - y' = a'(x_{,} - x') + b' + a'x' - y',$$

and combining it with the equation

$$y_{,} - y' = -\frac{1 + a'\cos.\omega}{a' + \cos.\omega}(x_{,} - x'),$$

the following results are obtained,

$$x_{,} - x' = \frac{(a' + \cos.\omega)(y' - a'x' - b')}{1 + a'^2 + 2a'\cos.\omega};$$

$$y_{,} - y' = -\frac{(1 + a'\cos.\omega)(y' - a'x' - b')}{1 + a'^2 + 2a'\cos.\omega}.$$

Substituting these values of $x_{,} - x'$, $y_{,} - y'$ in the formula, $\text{D} = \sqrt{\{(x_{,}-x')^2 + (y_{,}-y')^2 + 2(x_{,}-x')(y_{,}-y')\cos.\omega\}}$ (Art. 7.) and making the necessary reductions, it is found that

$$\text{D} = \pm\frac{(y' - a'x' - b')\sin.\omega}{\sqrt{\{1 + a'^2 + 2a'\cos.\omega\}}}.$$

CHAP. III.

TRANSFORMATION OF CO-ORDINATES.

19. THE general equation of a straight line is $y = \alpha x + b$, and the equation of a straight line which passes through the origin of the co-ordinates, $y = \alpha x$; the quantity denoted by α in both equations being different, according as the line is referred to rectangular or oblique axes, and more complicated in the case of the latter.

It is hence evident that the same line may be represented by an equation which is more or less simple, according as its position with respect to the axes is more or less simple, and also according as the axes are rectangular or oblique.

Consequently, if the position of a line on a plane is already determined by means of an equation, and it is observed that this line is in a more simple situation, with respect to two new straight lines than with respect to the primitive axes, it becomes of importance to deduce the equation of the line referred to the new axes, from the equation of the line referred to the primitive axes. This is the object proposed in the transformation of co-ordinates. The problem will be resolved if, for any point of the line, there are known the values of the primitive co-ordinates in functions of the new; for if these values are substituted in the equation proposed, the result gives a relation between the new co-ordinates of each of the points of the line under consideration.

Let OX, OY be the primitive axes, and $\text{OC} = x$, $\text{PC} = y$, the co-ordinates of any point P, referred to these axes: also let $\text{O}_{,}\text{X}_{,}$, $\text{O}_{,}\text{Y}_{,}$, be the new axes, and let $\text{O}_{,}\text{E} = x_{,}$, $\text{PE} = y_{,}$, be the co-ordinates of the same point P, referred to the new axes.

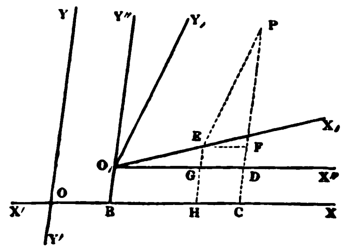
Through $\text{O}_{,}$ let $\text{BO}_{,}\text{Y}''$ be drawn parallel to OY, and $\text{O}_{,}\text{X}''$ parallel to OX; also through E let EGH be drawn parallel to OY, and EF parallel to OX.

Then there are given $\text{OB} = x'$, $\text{O}_{,}\text{B} = y'$, the co-ordinates of $\text{O}_{,}$, the new origin;

$\text{XOY} = \omega$, the angle made by the primitive axes;

$\text{X}''\text{O}_{,}\text{X}_{,} = \omega'$, the angle made by the primitive and the new axes of x; and

$X''O_{,}Y_{,}=\omega''$, the angle made by the primitive axis of x and the new axis of y; and that which is required is to express x and y, the primitive co-ordinates of any point P, in functions of the given quantities x', y', ω, ω', ω'', and of $x_{,}$, $y_{,}$, the new co-ordinates of the point P.



Because OX, $O_{,}X''$, EF are parallel lines, and OY, BY'', HE and CP are also parallel lines,

$X''O_{,}Y''=XOY=\omega$.

$O_{,}EG=EO_{,}Y''=X''O_{,}Y''-X''O_{,}X_{,}=\omega-\omega'$.

$O_{,}GE=180^\circ-X''O_{,}Y''=180^\circ-\omega$, and therefore $\sin. O_{,}GE=\sin. \omega$.

$FPE=Y_{,}O_{,}Y''=X''O_{,}Y''-X''O_{,}Y_{,}=\omega-\omega''$.

$PFE=O_{,}GE$ $\therefore \sin. PFE=\sin. O_{,}GE=\sin. \omega$.

1st, to find the expression for x,

$x=OC=OB+BH+HC$:

$OB=x'$;

$$BH=O_{,}G=\frac{O_{,}E \sin. O_{,}EG}{\sin. O_{,}GE}=\frac{x_{,} \sin. (\omega-\omega')}{\sin. \omega};$$

$$HC=EF=\frac{PE \sin. FPE}{\sin. PFE}=\frac{y_{,} \sin. (\omega-\omega'')}{\sin. \omega}$$

$$\therefore x=x'+\frac{x_{,} \sin. (\omega-\omega')+y_{,} \sin. (\omega-\omega'')}{\sin. \omega}$$

2nd, to find the expression for y.

$y=PC=CD+DF+FP$:

$CD=O_{,}B=y'$

$$DF=EG=\frac{O_{,}E \sin. X''O_{,}X_{,}}{\sin. O_{,}GE}=\frac{x_{,} \sin. \omega'}{\sin. \omega}$$

$$FP=\frac{PE \sin. PEF}{\sin. PFE}=\frac{PE \sin. X''O_{,}Y_{,}}{\sin. PFE}=\frac{y_{,} \sin. \omega''}{\sin. \omega}$$

$$\therefore y=y'+\frac{x_{,} \sin. \omega'+y_{,} \sin. \omega''}{\sin. \omega}.$$

$$x=x'+\frac{x_{,} \sin. (\omega-\omega')+y_{,} \sin. (\omega-\omega'')}{\sin. \omega},$$

$$y=y'+\frac{x_{,} \sin. \omega'+y_{,} \sin. \omega''}{\sin. \omega},$$

are the most general formulæ for the transformation of co-ordinates. They are not, however, so frequently employed

as other less general formulæ which are deducible from them, as particular cases.

20. Case I. Let the primitive axes be rectangular and the new axes oblique.

In this case $\omega = 90°$ $\therefore$ sin. ω = sin. $90° = 1$;

$$\sin.(\omega - \omega') = \sin.(90° - \omega') = \cos.\,\omega';$$
$$\sin.(\omega - \omega'') = \sin.(90° - \omega'') = \cos.\,\omega''$$
$$\therefore x = x' + x_{,} \cos.\,\omega' + y_{,} \cos.\,\omega''$$
$$y = y' + x_{,} \sin.\,\omega' + y_{,} \sin.\,\omega''.$$

21. Case II. Let the primitive axes be oblique and the new axes rectangular.

In this case $X_{,}OY_{,} = 90°$. $\therefore \omega'' = \omega' + 90°$

$$\therefore \omega - \omega'' = \omega - (\omega' + 90°) = -90° + (\omega - \omega')$$

and $\sin.(\omega - \omega'') = \sin.\big(-90° + (\omega - \omega')\big) = -\cos.(\omega - \omega')$
(Trigon. Art. 31.)
also $\sin.\,\omega'' = \sin.(\omega' + 90°) = \cos.\,\omega'$

$$\therefore x = x' + \frac{x_{,} \sin.(\omega - \omega') - y_{,} \cos.(\omega - \omega')}{\sin.\,\omega};$$

$$y = y' + \frac{x_{,} \sin.\,\omega' + y_{,} \cos.\,\omega'}{\sin.\,\omega}.$$

22. Case III. Let the primitive axes be rectangular, and the new axes also rectangular.

In this case, $\omega = 90°$; $\omega'' = 90° + \omega'$.

$$\sin.(\omega - \omega') = \sin.(90° - \omega') = \cos.\,\omega';$$
$$\sin.(\omega - \omega'') = \sin.\{90° - (90° + \omega')\} = \sin.(-\omega') = -\sin.\omega';$$
$$\sin.\,\omega = \sin.\,90° = 1;$$
$$\sin.\,\omega'' = \sin.(90° + \omega') = \cos.\,\omega';$$
$$\therefore x = x' + x_{,} \cos.\,\omega' - y_{,} \sin,\,\omega';$$
$$y = y' + x_{,} \sin.\,\omega' + y_{,} \cos.\,\omega'.$$

23. Case IV. Let the primitive and the new axes be parallel to each other.

In this case $\omega' = 0$ and $\omega'' = \omega$

$\therefore$ sin. $(\omega - \omega') = \sin.\,\omega$; sin. $(\omega - \omega'') = 0$; and sin. $\omega' = 0$

$$\therefore x = x' + \frac{x_{,} \sin.\,\omega}{\sin.\,\omega} = x' + x_{,}$$

$$y = y' + \frac{y_{,} \sin.\,\omega}{\sin.\,\omega} = y' + y_{,}.$$

24. Case V. Let the primitive and the new axes have the same origin, but different directions.

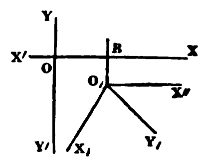
In this case $x'=0$ and $y'=0$

$$\therefore x = \frac{x_{,} \sin.(\omega-\omega')+y_{,} \sin.(\omega-\omega'')}{\sin.\omega};$$

$$y = \frac{x_{,} \sin.\omega' + y_{,} \sin. x''}{\sin.\omega}.$$

25. In applying the preceding formulæ for the transformation of co-ordinates to particular cases, it will be necessary to affect the co-ordinates with the signs of direction which belong to their positions, and also to give to the functions of the angles ω, ω', ω'', the signs which belong to them, in consequence of particular values of the angles, which, in order that the new may be capable of having every possible inclination to the primitive axes, may be any whatever between 0 and 360°.

As the origin of the new axes may be situated in any of the angles round the origin of the primitive axes, it will be also necessary to consider x', y' as capable of having positive or negative signs, according to their position with respect to the origin of the primitive axes.



As a particular case, illustrative of these observations,

Let $\text{OB}=x'=3$,
$\text{O}_{,}\text{B}=y'=2$,
$\text{XOY}=\omega=87°$
the reverse angle $\text{X}''\text{O}_{,}\text{X}_{,}=\omega'=240°$
the reverse angle $\text{X}''\text{O}_{,}\text{Y}_{,}=\omega''=315°$.

Substituting these values of x', y', ω, ω', ω'', in the general formulæ, and observing that from the position of $\text{O}_{,}\text{B}$, the sign of y' is negative,

$$x=3+\frac{x_{,} \sin.(87°-240°)+y_{,} \sin.(87°-315°)}{\sin. 87°}$$

$$y=-2+\frac{x_{,} \sin. 240°+y_{,} \sin. 315°}{\sin. 87°}$$

But (Trigon. Art. 19.),

$\sin.(87°-240°)=\sin.(-153°)=-\sin.(180°-153°)=-\sin. 27°$;
$\sin.(87°-315°)=\sin.(-228°)=-\sin.(180°-228°)=+\sin. 48°$;
$\sin. 240°=-\sin.(240°-180°)=-\sin. 60°$;
$\sin. 315°=-\sin.(360°-315°)=-\sin. 45°$.

Making these substitutions in the values of x, y,

$$x = 3 + \frac{-x_{,} \sin. 27° + y_{,} \sin. 48°}{\sin. 87°}$$

$$y = -2 - \frac{x_{,} \sin. 60° + y_{,} \sin. 45°}{\sin. 87°}.$$

26. Transformation of rectilinear into polar co-ordinates.

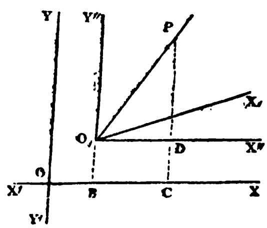
A line being traced on a plane, it may be proposed to determine a polar equation of that line, by assuming, in a convenient manner, the pole and the initial line.

Commonly, however, it is supposed that the equation of the line is already determined by means of an equation between the rectilinear co-ordinates of its points, and it is required from this equation to deduce another expressed in terms of polar co-ordinates.

To investigate general formulæ for the transformation of rectilinear into polar co-ordinates.

Let OC or x, and PC or y, be the co-ordinates of any point P, referred to the rectilinear axes OX, OY; also, let the angle $X_{,}O_{,}P$ or θ, and the radius vector $O_{,}P$, or r, be the polar co-ordinates of the same point.

Through $O_{,}$ let $O_{,}X''$, $O_{,}Y''$, be drawn parallel to OX, OY, respectively.



Then $XOY = \omega$, the angle of the rectilinear axes; $OB = x'$, $O_{,}B = y'$, the co-ordinates of the pole $O_{,}$; and $X''O_{,}X_{,} = \omega'$ the angle formed by the axis of x, and the initial line, are given:

and it is required to express x, y, in functions of ω', x', y', ω, r, and θ.

Because OX, $O_{,}X''$ are parallel lines, and OY, BY'', CP, are also parallel lines,

$X''O_{,}Y'' = XOY = \omega$,

$O_{,}PD = PO_{,}Y'' = X''O_{,}Y'' - X''O_{,}X_{,} - X_{,}O_{,}P = \omega - \omega' - \theta$,

$O_{,}DP = 180° - X''O_{,}Y'' = 180° - \omega$ $\therefore$ $\sin. O_{,}DP = \sin. \omega$.

$PO_{,}D = X''O_{,}X_{,} + X_{,}O_{,}P = \omega' + \theta$.

1st. To find the expression for x.

$x = OC = OB + BC$:

$OB = x'$

$$BC = O_{,}D = \frac{O_{,}P \sin. O_{,}PD}{\sin. O_{,}DP} = \frac{r \sin. (\omega - \omega' - \theta)}{\sin. \omega}$$

$$\therefore x = x' + \frac{r \sin. (\omega - \omega' - \theta)}{\sin. \omega}.$$

2nd. To find the expression for y.

$$y = \text{PC} = \text{CD} + \text{PD}:$$

$$\text{CD} = \text{O}_{,}\text{B} = y'$$

$$\text{PD} = \frac{\text{O}_{,}\text{P} \sin. \text{PO}_{,}\text{D}}{\sin. \text{O}_{,}\text{DP}} = \frac{r \sin. (\omega' + \theta)}{\sin. \omega}.$$

$$x = x' + \frac{r \sin. (\omega - \omega' - \theta)}{\sin. \omega},$$

$$y = y' + \frac{r \sin. (\omega' + \theta)}{\sin. \omega};$$

are the most general formulæ for the transformation of rectilinear into polar co-ordinates.

27. Particular cases.

Case I. Let the pole coincide with the origin of the rectilinear axes.

In this case $x' = 0$; $y' = 0$, and the formulæ become

$$x = \frac{r \sin. (\omega - \omega' - \theta)}{\sin. \omega},$$

$$y = \frac{r \sin. (\omega' + \theta)}{\sin. \omega}.$$

Case II. Let the initial line be parallel to the axis of x.

Since the initial line is parallel to the axis of x, the angle $\omega' = 0$,

$$\therefore x = x' + \frac{r \sin. (\omega - \theta)}{\sin. \omega}$$

$$y = y' + \frac{r \sin. \theta}{\sin. \omega}.$$

Case III. Let the pole coincide with the origin of the rectilinear axes and the initial line with the axis of x.

In this case, $x' = 0$; $y' = 0$; $\omega' = 0$

$$\therefore x = \frac{r \sin (\omega - \theta)}{\sin. \omega},$$

$$y = \frac{r \sin. \theta}{\sin. \omega}.$$

If the axes OX, OY, are rectangular,

$\sin. \omega = 1$,
$\sin. (\omega - \omega' - \theta) = \sin. (90° - (\omega' + \theta) = \cos. (\omega' + \theta)$,
$\sin. (\omega - \theta) = \sin. (90° - \theta) = \cos. \theta$,

The general formulæ become

Case IV.
$$x = x' + r \cos.(\omega' + \theta)$$
$$y = y' + r \sin.(\omega' + \theta).$$

And the formulæ of Cases I. II. III. become respectively,

Case V.
$$x = r \cos.(\omega' + \theta)$$
$$y = r \sin.(\omega' + \theta)$$

Case VI.
$$x = x' + r \cos.\theta;$$
$$y = y' + r \sin.\theta:$$

Case VII.
$$x = r \cos.\theta.$$
$$y = r \sin.\theta.$$

SECTION II.

NATURE AND PROPERTIES OF THE CONIC SECTIONS, OR CURVES OF THE SECOND ORDER.

CHAPTER I.

THE CONJUGATE DIAMETERS OF A CURVE OF THE SECOND ORDER.

28. UNDER the name of conic sections are comprehended the curve lines which arise from the intersections of a plane with the curve surface of an upright cone; the point, the straight line, the isosceles triangle, and the circle, which result respectively from the plane touching the cone at the vertex, or on the curve surface, or from cutting it through the vertex, or in a direction parallel to the base (Geom. Cor. Def. 7. Cylinder, &c.) not being, in general, included. But in the present section a general equation with two variable quantities is taken to denote a relation between the co-ordinates of any point in a curve line of the second order; and from such equation, which is of the second degree, the forms and principal properties of the curves will be analytically investigated. It will be shown in the third section that the curves are identical with those which result from the actual sections of a cone by a plane surface.

29. The general equation has usually the form

$$A'y'^2 + B'x'^2 + C'x'y' + D'y' + E'x' = F'. \qquad (A)$$

in which each of the coefficients, A′, B′, &c., represents some quantity differing in value in the different positions which the line or rectangle represented by it may have; but in any particular positions such coefficients are constant, while the co-ordinates x and y of any point in the curve, with reference to them, are variable. The co-ordinate axes may be supposed to form any angle, right or oblique, with one another.

30. This equation may, in the following manner, be transformed into one in which the origin of the co-ordinates shall have any position, and in which the co-ordinate axes shall make any angles with the original axes.

Let OX, OY (Fig. to Art. 19.) be the original co-ordinate axes, making with one another any angle; and let $O_{,}X_{,}$, $O_{,}Y_{,}$ be other axes, whose intersection is at any point $O_{,}$; also let $O_{,}X''$, $O_{,}Y''$ be axes passing through $O_{,}$ parallel to OX, OY. Again, let P be any point in a curve line of the second order, and let the ordinates PC, PE, EH be drawn as in the figure.

Now, if OB be represented by α, BO$'$ by β: also if $O_{,}E = x$, $EP = y$, $OC = x'$, $CP = y'$, the angle $XOY = \omega$, $X''O_{,}X_{,} = \omega'$, $X''O_{,}Y_{,} = \omega''$; and if p, q, r, t be put for

$$\frac{\sin.(\omega-\omega')}{\sin.\omega},\ \frac{\sin.\omega''}{\sin.\omega},\ \frac{\sin.\omega'}{\sin.\omega},\ \frac{\sin.(\omega-\omega'')}{\sin.\omega}$$

respectively: then the equations for x and y (Art. 19.) will become

$$x' = \alpha + px + ty\ (= \text{OC}),$$
$$y' = \beta + rx + qy\ (= \text{CP}).$$

Substituting these values in the equation (A) the latter becomes, after arranging the terms according to the powers of y and x,

$$\begin{aligned}&(A'q^2 + B't^2 + C'qt)\,y^2\\ &+(A'r^2 + B'p^2 + C'pr)\,x^2\\ &+(2qrA' + 2ptB' + pqC' + rtC')\,xy \qquad\qquad (A')\\ &+(2\beta qA' + 2\alpha tB' + \alpha qC' + \beta tC' + qD' + tE')\,y\\ &+(2\beta rA' + 2\alpha pB' + \alpha rC' + \beta pC' + rD' + pE')\,x\\ &= F -, \&c.\end{aligned}$$

the second member containing all the terms which are not multiplied by the variable x or y. This is the required equation, and it is manifestly of the form

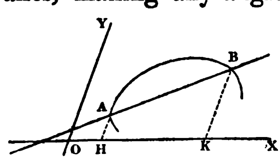
$$Ay^2 + Bx^2 + Cxy + Dy + Ex = F. \qquad (A'')$$

It coincides with the original equation (A) when α, β, ω', are each zero, and when $\omega = \omega''$; in which case $p = 1$, $q = 1$, $r = 0$, $t = 0$.

PROPOSITION I.

31. A straight line can cut a curve which is designated by an equation of the second degree in two points only.

Let OX, OY be two co-ordinate axes, making any angle with one another, and let AB be a straight line cutting the curve; the equation of this line may (Art. 10.) be expressed by



$$y = Mx + N:$$

now at the point or points in which AB cuts the curve the co-ordinates x and y of the straight line will be identical with x

and y in the equation of the curve; therefore this value of y being substituted in the equation (A) or (A''), the result will be an equation containing only the first and second powers of x with constants: it may, consequently, be put in the form

$$\text{P}x^2+\text{Q}x=\text{T}.$$

This equation being resolved, the two values of x, if not imaginary, will denote the distances OH, OK of the ordinates which pass through the points of intersection; and since a quadratic equation can have but two real roots, the straight line can cut the curve in two points only.

If the two values of x in the equation should be equal to one another, and have the same sign, the straight line would meet the curve in one point only.

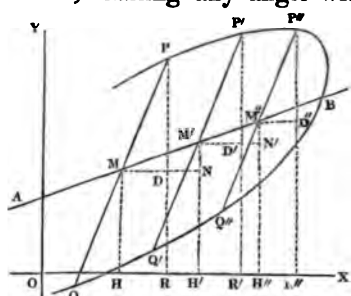
32. DEFINITION 1. Any straight line which is terminated at each extremity in a curve line, is called a chord of the curve.

33. DEF. 2. Any straight line which meets a curve line in one point only, is called a tangent of the curve.

PROPOSITION II.

34. If a straight line be drawn so as to bisect two chords which are parallel to one another in a curve of the second order, it will bisect all the chords which may be drawn parallel to these.

Let OX, OY be co-ordinate axes, making any angle with one another: let P′Q′, P″Q″ be two chords parallel to each other, and let them be bisected in M′ and M″ by the straight line AB; it may be proved in the following manner that AB will bisect any other chord as PQ parallel to the former chords:—



Let P′M′, M′Q′ be represented respectively by $+k'$ and $-k'$; P″M″, M″Q″ by $+k''$ and $-k''$; also, imagining AB to intersect PQ in M, let PM be represented by $+k$: again, the ordinates PR, P′R′, &c., MH, M′H′, &c. being drawn parallel to OY, through M draw MN cutting PR in D, and through M′ draw M′N′, each parallel to OX. Then, in the triangle MDP, putting v for $\dfrac{\sin.\ \text{PMD}}{\sin.\ \text{PDM}}$, and u for $\dfrac{\sin.\ \text{MPD}}{\sin.\ \text{PDM}}$, we have (Trigon. Art. 57.)

$$\text{PD}=vk,\ \text{DM}=uk,$$

and corresponding values will express the sides of the triangles $\mathrm{P'M'D'}$, $\mathrm{P''M''D''}$.

Now, let the co-ordinates of M, $\mathrm{M'}$, $\mathrm{M''}$ be as follow:—

$$\mathrm{OH}=\gamma,\ \mathrm{OH'}=\gamma',\ \mathrm{OH''}=\gamma''$$
$$\mathrm{HM}=\delta,\ \mathrm{H'M'}=\delta',\ \mathrm{H''M''}=\delta'';$$

then the co-ordinates of the points P, $\mathrm{P'}$, $\mathrm{P''}$, with respect to OX, and OY will become

$$\mathrm{OR}=\gamma+uk,\qquad \mathrm{RP}=\delta+vk,$$
$$\mathrm{OR'}=\gamma'+uk',\quad \mathrm{R'P'}=\delta'+vk',$$
$$\mathrm{OR''}=\gamma''+uk'',\ \mathrm{R''P''}=\delta''+vk''.$$

Substituting the co-ordinates of $\mathrm{P'}$ in place of x' and y' in the equation (A) the latter will become, for the point $\mathrm{P'}$,

$$\mathrm{A'}(\delta'+vk')^2+\mathrm{B'}(\gamma'+uk')^2+\mathrm{C'}(\delta'+vk')(\gamma'+uk')+\mathrm{D'}(\delta'+vk')+\mathrm{E'}(\gamma'+uk')=\mathrm{F};\ . \qquad (a)$$

and placing double accents on γ, δ, and k, a like equation will be obtained for the point $\mathrm{P''}$.

In the two equations thus formed let k' and k'' be the quantities to be determined; then, since the equations are quadratic, and that the two values of each of the roots k' and k'' are, by hypothesis, equal to one another, with contrary signs; it follows, from the nature of such equations (Algebra, Art. 121.), that the coefficients of k' and k'' must be respectively zero.

Hence, collecting all the terms which contain the first powers of k' and k'', we have

$$2\,\delta' v\mathrm{A'}+2\,\gamma' u\mathrm{B'}+\gamma' v\mathrm{C'}+\delta' u\mathrm{C'}+v\mathrm{D'}+u\mathrm{E'}=0$$

and $\qquad (b)$

$$2\,\delta'' v\mathrm{A'}+2\,\gamma'' u\mathrm{B'}+\gamma'' v\mathrm{C'}+\delta'' u\mathrm{C'}+v\mathrm{D'}+u\mathrm{E'}=0$$

Subtracting the first of these equations from the second, there results

$$2v\mathrm{A'}(\delta''-\delta')+2\,u\mathrm{B'}(\gamma''-\gamma')+v\mathrm{C'}(\gamma''-\gamma')+u\mathrm{C'}(\delta''-\delta')=0;$$

whence $$\frac{\gamma''-\gamma'}{\delta''-\delta'}=-\frac{u\mathrm{C'}+2\,v\mathrm{A'}}{v\mathrm{C'}+2\,u\mathrm{B'}}:$$

or, from the similarity of the triangles $\mathrm{MNM'}$, $\mathrm{M'N'M''}$,

$$\frac{\gamma'-\gamma}{\delta'-\delta}\left(=\frac{\gamma''-\gamma'}{\delta''-\delta'}\right)=-\frac{u\mathrm{C'}+2v\mathrm{A'}}{v\mathrm{C'}+2u\mathrm{B'}};$$

therefore

$$(v\mathrm{C'}+2\,u\mathrm{B'})\,\gamma'+(u\,\mathrm{C'}+2v\mathrm{A'})\,\delta'=(u\mathrm{C'}+2\,v\mathrm{A'})\,\delta+(v\,\mathrm{C'}+2\,u\mathrm{B'})\,\gamma.$$

Now, imagining the co-ordinates $\gamma+uk$ and $\delta+vk$ of the point P to be substituted respectively for x' and y' in the equation (A) the result will be identical with the equation (a), omitting the accents on γ, δ, and k in the latter; and conceiv-

ing this result to be resolved as a quadratic equation, in which k is the unknown quantity, the sum of the coefficients of the first power of k will have for its numerator (conformably to the first member of either of the equations (b)),

$$2\,\delta v A' + 2\,\gamma u B' + \gamma v C' + \delta u C' + v D' + u E',$$

the common denominator being the sum of the co-efficients of k^2.

But the four first terms in this expression constitute the second member of the last equation above; therefore those four terms are equivalent to its first member: and, on comparing the expression (after substituting in it that equivalent) with the first of the equations (b), it will be seen that the expression is zero. The coefficients of the first power of k being together zero, it follows that the two values of k in the equation (A) are equal to one another with contrary signs: that is, $PM = MQ$, or the chord PQ is bisected in M by the line AB. In like manner all other chords parallel to $P'Q'$ may be shown to bisected by AB.

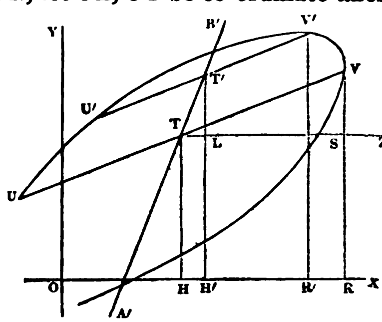
35. DEF. 3. A straight line bisecting all the chords which are parallel to one another, is called a diameter of the curve.

36. DEF. 4. If any number of chords be drawn parallel to a diameter, the straight line bisecting them (which is consequently a diameter) is said to be a diameter conjugate to the former.

PROPOSITION III.

37. If a diameter be conjugate to another, it will be parallel to the chords which are bisected by the latter diameter.

As in the figure to Art. 34., let OX, OY be co-ordinate axes, making any angle with one another; and let UV, $U'V'$ be two chords parallel to AB in that figure; also, let $A'B'$ be the diameter which bisects them: it may be proved in the following manner that $A'B'$ is parallel to PQ:—



Imagine the ordinates TH, $T'H'$, VR, $V'R'$ to be drawn parallel to OY, and the line LS through T parallel to OX. Let

$$OH = \mu,\ OH' = \mu',\ HT = \nu,\ H'T' = \nu';$$

and in the triangle TVS putting v' for $\frac{\sin. \text{VTS}}{\sin. \text{TSV}}$ and u' for $\frac{\sin. \text{TVS}}{\sin. \text{TSV}}$; also representing TV, T′V′ by l and l', we have

$$\text{OR} = \mu + u'l, \quad \text{RV} = \nu + v'l,$$
$$\text{OR}' = \mu' + u'l', \quad \text{R}'\text{V}' = \nu' + v'l'.$$

Substituting these values of OR, RV, and again those of OR′, R′V′, in place of x' and y' in the equation (A); and from the two resulting equations collecting the coefficients of the first power of l and of l': then making the sums of the coefficients separately equal to zero (since the positive and negative values of l and l' are, respectively, equal to one another with contrary signs), and proceeding exactly as in Art. 34., there will be obtained

$$\frac{\mu'-\mu}{\nu'-\nu} = -\frac{u'\text{C}'+2v'\text{A}'}{v'\text{C}'+2u'\text{B}'}.$$

But the triangle TSV in the present figure is, from the parallelism of the lines, similar to MNM′ (Fig. to Art. 34.); therefore

$$\text{TS} : \text{SV} :: \text{MN} : \text{NM}'$$

or $$u'l : v'l :: \gamma'-\gamma : \delta'-\delta;$$

whence $$\frac{u'}{v'} = \frac{\gamma'-\gamma}{\delta'-\delta}.$$

Substituting for the last fraction its equivalent (Art. 34.) we have

$$\frac{u'}{v'} = -\frac{u\text{C}'+2v\text{A}'}{v\text{C}'+2u\text{B}'}; \qquad (c)$$

or $$2uu'\text{B}'+2vv'\text{A}' = -(uv'+vu')\,\text{C}'; \qquad (d)$$

whence $$\frac{u}{v} = -\frac{u'\text{C}'+2v'\text{A}'}{v'\text{C}'+2u'\text{B}'}. \qquad (e)$$

Thus $$\frac{u}{v} = \frac{\mu'-\mu}{\nu'-\nu};$$

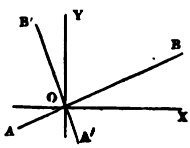
that is, $\frac{\text{MD}}{\text{DP}} = \frac{\text{TL}}{\text{LT}'}$, or PQ and A′B′ make equal angles with the axis OX or OY; consequently A′B′ is parallel to PQ.

38. COROLLARY. Hence, when two diameters are conjugate to one another, each of them is parallel to the chords which are bisected by the other; and each diameter bisects the other. Also, since the segments, as PM, MQ, or UT, TV, of every chord which is bisected by the same diameter are constantly equal to one another, they must vanish together at each

extremity of the diameter; and a line drawn at each extremity, as B, of a diameter, parallel to the bisected chords PQ (Fig. to Art. 34.) would meet the curve in that point only: such line is, consequently, a tangent to the curve at that point. It follows that the tangents at the opposite extremities of a diameter are parallel to the bisected chords, to the conjugate diameter, and to one another.

39. *Scholium.* Since the parallel chords PQ, &c. may make any angles with the co-ordinate axes, it is evident that there may be an infinite number of pairs of conjugate diameters, AB, A′B′, making different angles with one another; and it may be proved that among these pairs there is only one in which the diameters intersect one another at right angles.

Thus, let OX, OY be co-ordinate axes at right angles to one another, and let AB, A′B′, which, for simplicity, may be supposed to intersect one another in O, represent two diameters also at right angles to one another. Then, the angles corresponding to PDM and TSV (Figs. to Arts. 34. and 37.) being right angles, v' = sin. XOB and u' = cos. XOB; also v = sin. XOB′ and $-u$ = cos. XOB′; therefore



$$\frac{v'}{u'} = \text{tan. XOB},$$

$$\text{and} -\frac{v}{u} = -\text{tan. XOB}'.$$

But the angle XOB′ = 90° + XOB; therefore

$$-\text{ tan. XOB}' = \text{cotan. XOB},$$

and representing

$\frac{v'}{u'}$ by q, we have (Trigon. Art. 27.) cotan. XOB $= \frac{1}{q}$:

whence $$-\frac{v}{u} = \frac{1}{q}, \text{ or } \frac{u}{v} = -q.$$

Now (c) (d) and (e) are, each of them, an equation relating to two conjugate diameters; and dividing every term in the second member of the equation (c) by v, the result after substituting—q for $\frac{u}{v}$ may be put in the form

$$q = -\frac{\text{C}' - 2q\text{B}'}{2\text{A}' - q\text{C}'}, \text{ or } q = \frac{\text{C}' - 2q\text{B}'}{q\text{C}' - 2\text{A}'};$$

whence, after reduction,

$$q^2 - \frac{2\,(\mathrm{A}'-\mathrm{B}')}{\mathrm{C}'}\,q - 1 = 0:$$

transposing, completing the square, &c., we get

$$q = -\frac{\mathrm{A}'-\mathrm{B}}{\mathrm{C}'} \pm \surd\left\{1 + \left(\frac{\mathrm{A}'-\mathrm{B}'}{\mathrm{C}'}\right)^2\right\}.$$

The quantity under the radical sign is necessarily positive; therefore the roots, or values of q, are real, which proves that a pair of conjugate diameters may be at right angles to one another. But the equation for zero being quadratic, there can be only two roots; and the third term (which is equivalent to the product of the roots) being -1, the roots must be reciprocals of one another, one positive and the other negative; so that if one of them be q, the other is $-\frac{1}{q}$. Hence the two angles which one of the diameters may make with OX may be expressed by ω and $90°+\omega$; so that if AB be one of the positions, the other position will coincide with that of A′B′, on the opposite side of OY. Thus there can be only one pair of conjugate diameters at right angles to one another.

40. DEF. 5. Two conjugate diameters at right angles to one another are called the axes of the curve: the longer is called the *transverse*, or *major axis*, and the shorter the *conjugate*, or *minor axis*. Either extremity of the transverse axis is frequently called the vertex of the curve.

CHAP. II.

MODIFICATIONS OF THE GENERAL EQUATION OF THE SECOND DEGREE.

41. In the general equation (A′) or (A″) there are five arbitrary constants α, β, ω, ω', and ω'', on the values of which depend the positions of the co-ordinate axes $O_{,}X_{,}$, $O_{,}Y_{,}$ (Fig. to Art. 19.): these equations may therefore be simplified by making the coefficients of some of the terms vanish, provided the arbitrary constants be supposed to have the values derived from the equations which result on such coefficients being assumed equal to zero. It may be observed also, that the species of a curve of the second order will vary according to the relative magnitudes of some of the coefficients in (A″), and according as the signs of some of those coefficients are alike or contrary.

Let the equation (A″) be reduced to

$$Ay^2 + Bx^2 + Cxy = F, \qquad (1)$$

or let the coefficients of y and x in the equation (A′) be separately zero; we shall then have

$$2\beta q A' + 2\alpha t B' + \alpha q C' + \beta t C' + q D' + t E' = 0$$

and $$2\beta r A' + 2\alpha p B' + \alpha r C' + \beta p C' + r D' + p E' = 0.$$

Multiplying the first of these equations by p, and the second by t, and taking one result from the other: again, multiplying the first by r, and the second by q, and subtracting the first result from the second, there will be obtained

$$2\beta A' + \alpha C' + D' = 0,$$

and $$2\alpha B' + \beta C' + E' = 0.$$

From these equations we get

$$\alpha = \frac{C'D' - 2A'E'}{4A'B' - C'^2}, \text{ and } \beta = \frac{E'C' - 2B'D'}{4A'B' - C'^2},$$

which are the values of the co-ordinates OB and $BO_{,}$ of the point $O_{,}$ with respect to the original axes OX, OY. (Fig. to Art. 19.)

Now $O_{,}X_{,}$ and $O_{,}Y_{,}$ may be the directions of any two dia-

meters of the curve, intersecting (and bisecting) one another in $O_{\prime}$; therefore the values of α and β become those of the co-ordinates of such point of section. The terms which enter into those values being constant; it follows that, except when α and β are infinite, all the diameters of the curve intersect one another in one point, which is their common point of bisection.

a. If $\alpha=0$, and $\beta=0$, the co-ordinate axes OX, OY coincide with $O_{\prime}X''$, $O_{\prime}Y''$. If α and β are infinite, in which case $4A'B'=C'^2$, all the diameters of the curve will be parallel to one another.

42. DEF. 6. The point in which all the diameters intersect one another is called the centre of the curve.

a. Scholium. The equation (I) appertains to a curve of the second order, which has a centre, the origin $O_{\prime}$ of the co-ordinates being, when $\alpha=0$ and $\beta=0$, coincident with the centre; but, since ω' and ω'' remain arbitrary, the co-ordinate axes $O_{\prime}X_{\prime}$, $O_{\prime}Y_{\prime}$, or the diameters with which they coincide, may make any angle with one another.

43. Let the equation (A$''$) be reduced to

$$Ay^2+Bx^2+Dy+Ex=F, \qquad \text{(II)}$$

or let the sum of the coefficients of xy in the equation (A$'$) be zero: we then get

$$2qrA'+2ptB'+pqC'+rtC'=0.$$

In this equation, substituting the values of p, q, r, t (Art. 30.), omitting the denominators, which are common to all the terms, we obtain

$$2A'\sin.\omega''\sin.\omega'+2B'\sin.(\omega-\omega')\sin.(\omega-\omega'')+\{\sin.(\omega-\omega')\sin.\omega''+\sin.\omega'\sin.(\omega-\omega'')\}C'=0.$$

In which ω' and ω'' are respectively the angles which $O_{\prime}X_{\prime}$ and $O_{\prime}Y_{\prime}$ make with OX or $O_{\prime}X''$, while ω is the angle XOY or $X''O_{\prime}Y''$. (Fig. to Art. 19.)

Now if $O_{\prime}X_{\prime}$ be supposed to correspond to AB or TV, and $O_{\prime}Y_{\prime}$ to A$'$B$'$ or MP (Figs. to Arts. 34. and 37.), we shall have

BMD, or VTS$=\omega'$, and PMD$=\omega''$; also, since PDN or VSZ$=\omega$, TVS$=\omega-\omega'$, and MPD$=\omega-\omega''$:

therefore, if every term in the last equation for zero be divided by $\sin.^2\omega$, on substituting for $\dfrac{\sin.\omega'}{\sin.\omega}\left(=\dfrac{\sin.\text{VTS}}{\sin.\text{VSZ}}\right)$,

and for $\dfrac{\sin.(\omega-\omega')}{\sin.\omega}\left(=\dfrac{\sin.\text{TVS}}{\sin.\text{VSZ}}\right)$, (Art. 37.)

their respective equivalents v' and u'; also,

for $\frac{\sin. \omega''}{\sin. \omega}\left(=\frac{\sin. \text{PMD}}{\sin. \text{PDN}}\right)$, and for $\frac{\sin. (\omega-\omega'')}{\sin. \omega}\left(=\frac{\sin. \text{MPD}}{\sin. \text{PDN}}\right)$, (Art. 34.)

their respective equivalents v and u, that equation becomes

$$2\text{A}'v'v+2\text{B}'u'u+\text{C}'(u'v+v'u)=0,$$

which is identical with the equation (d), Art. 37.; and consequently it relates to two conjugate diameters. It follows that when, in the general equation (A''), the coefficient of xy is equivalent to zero, the co-ordinate axes, as $\text{O}_{,}\text{X}_{,}$, $\text{O}_{,}\text{Y}_{,}$, make angles with OX, OY equal to those which are made, with the same axes, by two conjugate diameters; or $\text{O}_{,}\text{X}_{,}$, $\text{O}_{,}\text{Y}_{,}$ are parallel to two conjugate diameters of the curve.

44. Let the equation (A'') be reduced to

$$\text{A}y^2+\text{B}x^2+\text{D}y=\text{F}; \qquad \text{(III)}$$

then, since $\text{C}=0$, the co-ordinate axes $\text{O}_{,}\text{X}_{,}$, $\text{O}_{,}\text{Y}_{,}$ will (Art. 43.) be parallel to two conjugate diameters; and, since $\text{E}=0$, the sum of the co-efficients of x in (A') is zero: whence we have

$$2\beta r\text{A}'+2\alpha p\text{B}'+\alpha r\text{C}'+\beta p\text{C}'+r\text{D}'+p\text{E}'=0,$$

or $2\beta\text{A}'\sin. \omega'+2\alpha\text{B}'\sin. (\omega-\omega')+\alpha\text{C}'\sin. \omega'+\beta\text{C}'\sin. (\omega-\omega')+\text{D}'\sin. \omega'+\text{E}'\sin. (\omega-\omega')=0.$

This equation may be verified on substituting for β its value (Art. 41.), and making $\alpha=0$, or $\text{C}'\text{D}'=2\text{A}'\text{E}'$; all the terms being brought to the common denominator $4\text{A}'\text{B}'-\text{C}'^2$, and the necessary reductions made. It follows, therefore, that the equation (III) appertains to the curve when the co-ordinate axis $\text{O}_{,}\text{Y}_{,}$ is coincident in direction with a diameter, and $\text{O}_{,}\text{X}_{,}$ is parallel to that which is conjugate to it.

a. In a similar manner, if the equation (A'') be reduced to

$$\text{A}y^2+\text{B}x^2+\text{E}x=\text{F}, \qquad \text{(IV)}$$

it may be shown that this last appertains to the curve when the co-ordinate axis $\text{O}_{,}\text{X}_{,}$ is coincident in direction with a diameter, and $\text{O}_{,}\text{Y}_{,}$ is parallel to that which is conjugate to it.

For D being zero, on making the coefficient of y in (A') equal to zero, substituting in the result $\sin. \omega''$ for q, and for t, $\sin. (\omega-\omega'')$, the equation may be verified on substituting for α its value (Art. 41.), and making $\beta=0$, or $\text{E}'\text{C}'=2\text{B}'\text{D}'$.

45. Let the equation (A'') be reduced to

$$\text{A}y^2+\text{B}x^2=\text{F}. \qquad \text{(V)}$$

Then, since $\text{D}=0$ and $\text{E}=0$, on making the coefficients of y and x in (A') separately zero, the resulting equations will be verified by making $\alpha=0$, $\beta=0$: for, after cancelling the terms which are thus destroyed, we have

$$\text{D}'\sin. \omega''+\text{E}'\sin. (\omega-\omega'')=0,$$

and $$\text{D}'\sin. \omega'+\text{E}'\sin. (\omega-\omega')=0;$$

whence, ω' and ω'' being unequal to one another, $D'=0$, and $E'=0$. Therefore, since $\alpha=0$, $\beta=0$, and $C=0$, the co-ordinate axes $O_,X_{,,}$ $O_,Y_,$ are (Arts. 42. *a.* and 43.) coincident with two conjugate diameters.

46. Let the equation (A'') be reduced to

$$Ay^2+Dy+Ex=F, \qquad \text{(VI)}$$

in which both B and C, or the coefficients of x^2 and of xy in (A'), are zero.

From the equation $A'r^2+B'p^2+C'pr=0$

we have $$C'=-\frac{A'r^2+B'p^2}{pr};$$

and substituting this value of C' in the equation

$$2qrA'+2ptB'+pqC'+rtC'=0,$$

we have, after reduction,

$$B'p^2=A'r^2; \text{ whence } B'=\frac{A'r^2}{p^2},$$

$$\text{or } 4A'B'=4A'^2\frac{r^2}{p^2}.$$

Putting $A'r^2$ for $B'p^2$ in the above value of C', we get

$$C'=-\frac{2A'r}{p};$$

therefore $C'^2=4A'B'$, or $4A'B'-C'^2=0$:
consequently, all the diameters of the curve will (Art. 41. *a.*) be parallel to one another.

The value of C, or the coefficient of xy in (A'), being zero; it would follow (Art. 43.), if the curve had a centre, that the co-ordinate axes $O_,X_{,,}$ $O_,Y_,$ were parallel to two conjugate diameters; but all the diameters being, in the present case, parallel to one another, one of the axes will be parallel to the diameters, and the other parallel to the chords which are bisected by that diameter.

a. If the equation (A'') be reduced to

$$Ay^2+Ex=F; \qquad \text{(VII)}$$

since then $D=0$, it will follow (Art. 44. *a.*) that the co-ordinate axis is coincident in direction with that particular diameter whose bisected chords are parallel to $O_,Y_,$.

47. By a modification of the general equation (A') or (A''), Art. 30., there may be obtained the conditions under which the coefficients of the squares of the variable quantities x and y have like, or unlike signs. Thus, since both ω' and ω'' are arbitrary, let, first, $C=0$ in the equation (A''), in which case

(Art. 43.) the co-ordinate axes $\mathrm{O}_{,}\mathrm{X}_{,}$, $\mathrm{O}_{,}\mathrm{Y}_{,}$ are parallel to two conjugate diameters of the curve: we have then, as in that article,

$$2\mathrm{A}'\sin.\omega''\sin.\omega'+2\mathrm{B}'\sin.(\omega-\omega')\sin.(\omega-\omega'')=-\mathrm{C}'(\sin.(\omega-\omega')\sin.\omega''+\sin.\omega\sin.(\omega-\omega'')).$$

But supposing, for simplicity, that O X, O, Y are at right angles to one another, in which case $\omega=90^\circ$, we have

$$2\mathrm{A}'\sin.\omega''\sin.\omega'+2\mathrm{B}'\cos.\omega''\cos.\omega'=-\mathrm{C}'(\cos.\omega'.\sin.\omega''+\sin.\omega'\cos.\omega'').$$

Again let $\omega''=90^\circ+\omega'$, in which case the axes $\mathrm{O}_{,}\mathrm{X}_{,}$, $\mathrm{O}_{,}\mathrm{Y}_{,}$ are also at right angles to one another; then $\sin.\omega''=\cos.\omega'$ and $\cos.\omega''=-\sin.\omega'$, and the last equation becomes

$$2(\mathrm{A}'-\mathrm{B}')\sin.\omega'\cos.\omega'=\mathrm{C}'(\sin.^2\omega'-\cos.^2\omega');$$

hence, $2\sin.\omega'\cos.\omega'=\dfrac{\mathrm{C}'}{\mathrm{A}'-\mathrm{B}'}(\sin.^2\omega'-\cos.^2\omega')$,

or (Trigon. Arts. 34. 34. *a*.),

$$\sin.2\omega'=-\frac{\mathrm{C}'}{\mathrm{A}'-\mathrm{B}'}\cos.2\omega';\ \text{or again, }\sin.2\omega'=\frac{\mathrm{C}'}{\mathrm{B}'-\mathrm{A}'}\cos.2\omega';$$

therefore (Trigon. Art. 20. *b*.),

$$\tan.2\omega'=\frac{\mathrm{C}'}{\mathrm{B}'-\mathrm{A}'}\text{ and }\mathrm{cotan}.2\omega'=\frac{\mathrm{B}'-\mathrm{A}'}{\mathrm{C}'}.$$

Also, $\cos.^2 2\omega'\left(=\text{Trigon. Art. 24. }e,\dfrac{1}{1+\tan.^2 2\omega'}\right)=\dfrac{(\mathrm{B}'-\mathrm{A}')^2}{(\mathrm{B}'-\mathrm{A}')^2+\mathrm{C}'^2}$,

and $\sin.^2 2\omega'\left(=\text{Trigon. Art. 18. }b,\dfrac{1}{1+\mathrm{cotan}.^2 2\omega'}\right)=\dfrac{\mathrm{C}'^2}{(\mathrm{B}'-\mathrm{A}')^2+\mathrm{C}'^2}$.

Now with the above values of $\sin.\omega''$, $\cos.\omega''$ the equivalent of A in (A″) becomes (the denominators in the values of p, q, r, and t being unity)

$$\mathrm{A}'\cos.^2\omega'+\mathrm{B}'\sin.^2\omega'-\mathrm{C}'\sin.\omega'\cos.\omega',$$

or (Trigon. Art. 35.),

$$\tfrac{1}{2}\mathrm{A}'(1+\cos.2\omega')+\tfrac{1}{2}\mathrm{B}'(1-\cos.2\omega')-\tfrac{1}{2}\mathrm{C}'\sin.2\omega',\quad(f)$$

while that of B is

$$\mathrm{A}'\sin.^2\omega'+\mathrm{B}'\cos.^2\omega'+\mathrm{C}'\sin.\omega'\cos.\omega'$$

or, $\tfrac{1}{2}\mathrm{A}'(1-\cos.2\omega')+\tfrac{1}{2}\mathrm{B}'(1+\cos.2\omega')+\tfrac{1}{2}\mathrm{C}'\sin.2\omega'.\quad(g)$

But $$1+\cos.2\omega'=1+\frac{\mathrm{B}'-\mathrm{A}'}{\sqrt{\{(\mathrm{B}'-\mathrm{A}')^2+\mathrm{C}'^2\}}},$$

$$1-\cos.2\omega'=1-\frac{\mathrm{B}'-\mathrm{A}'}{\sqrt{\{(\mathrm{B}'-\mathrm{A}')^2+\mathrm{C}'^2\}}},$$

and $$\sin.2\omega'=\frac{\mathrm{C}'}{\sqrt{\{(\mathrm{B}'-\mathrm{A}')^2+\mathrm{C}'^2\}}}$$

therefore, multiplying the two terms on the right hand of the first equation by $\tfrac{1}{2}\mathrm{A}'$, the two terms on the right hand of the

second by $\frac{1}{2}$ B$'$, and the right hand of the third by $-\frac{1}{2}$ C$'$: adding the results together and simplifying, we obtain from the equation (f), A (the coefficient of y^2 in (A$''$))

$$= \tfrac{1}{2}(A'+B') - \tfrac{1}{2}\sqrt{\{(B'-A')^2+C'^2\}}.$$

Proceeding in like manner we obtain, from (g), B (the coefficient of x^2 in (A$''$))

$$= \tfrac{1}{2}(A'+B') + \tfrac{1}{2}\sqrt{\{(B'-A')^2+C'^2\}}.$$

It follows that A and B are real quantities: it follows also that, while A$'$ and B$'$ have positive signs, the value of B is necessarily positive, and that A is positive only when

$$A'+B' \text{ is greater than } \sqrt{\{(B'-A')^2+C'^2\}},$$

or $A'^2+2A'B'+B'^2$ is greater than $A'^2-2A'B'+B'^2+C'^2$; that is, when $2A'B'$ is greater than $C'^2-2A'B'$,

or when $4A'B'$ is greater than C'^2.

a. Hence when $4A'B' > C'^2$, the coefficients of y^2 and x^2 in (A$'$) have like signs, which are positive or may be rendered so by changing the signs of all the terms in the equation (A); and when $4A'B' < C'^2$, the coefficients of y^2 and x^2 in (A$'$) have unlike signs.

If A$'$ and B$'$ have contrary signs, in which case both $4A'B'$ and C'^2 are negative in the expression $4A'B'-C'^2$; it will be found, on performing the like operations, that A and B have also contrary signs.

48. On resolving the equation (A) Art. 29. considering y' as the only unknown quantity, we shall have

$$y' = -\frac{C'x'+D'}{2A'} \pm \frac{1}{2A'}\sqrt{\{(C'^2-4A'B')\,x'^2+(2C'D'-4A'E')\,x'+D'^2+4A'F'\}}:$$

now, if A and B in (A$''$) Art. 30. have like signs, A$'$ and B$'$ will also have like signs; and $4A'B'$ being then greater than C'^2, the coefficient of x'^2 in the above value of y' is negative; therefore, if x' were increased indefinitely, the sum of the negative quantities under the radical sign would exceed that of the positive quantities; hence it follows that the value of x', both positive and negative, is limited to a certain quantity, beyond which the value of y' would become imaginary; and thus the curve is limited in extent on opposite sides of the point at which x' is zero. If the equation (A) were resolved in like manner with respect to x', we should have $C'^2-4A'B'$ for the coefficient of y'^2 under the radical sign; and thus it is evident that the curve is limited in extent in every direction on opposite sides of the point at which y' is zero.

a. DEF. 7. A curve of the second order, which is limited

on opposite sides of the point at which both x' and y' are zero, is called an ellipse.

b. If A and B in (A″) Art. 30. have unlike signs, we have, as above, 4 A′B′ less than C'^2, and consequently the coefficient of x'^2, in the above value of y', is positive; it follows, therefore, that while x', whether positive or negative, has any value beyond a certain finite quantity, the value of y' cannot become imaginary. Thus the curve may extend to an infinite distance on each side of the point at which x' is zero. If the equation (A) Art. 29. be resolved with respect to x', it will be found, in like manner, that the curve may extend to an infinite distance on each side of the point at which y' is zero.

c. DEF. 8. A curve of the second order, which is unlimited in extent on opposite sides of the point at which both x' and y' are zero, is called an hyperbola.

d. When $4\,A'B'=C'^2$, the co-ordinates α and β (Art. 41.) become infinite, or the curve has no centre: in this case the term $(C'^2-4\,A'B')\,x'^2$, in the above value of y', vanishes; and ($C'D'-2\,A'E'$ being considered as positive) the value of y' cannot become imaginary while x' is positive, and when it exceeds a certain finite quantity. If x', being negative, were to exceed a certain finite quantity, y' would be imaginary; thus the curve is infinite in one direction only.

e. DEF. 9. A curve of the second order, which may extend to infinity in one direction only, is called a parabola.

49. From the value of y' in the preceding article it is evident that, when $F'=0$ in the equation (A), if x' be made zero, the value of y' will vanish: it follows that, in the general equation (A′) or (A″) Art. 30., and in any of the equations from (I) to (VII) inclusive, when $F=0$ if, at the same time, $x=0$, the equation will be satisfied by making $y=0$, or by placing the origin of the co-ordinates on the curve line.

50. When in the equation (V) the coefficients A and B are both positive (which is the condition for an ellipse), should F be negative the equation would be absurd, and would lead to no result. Should $F=0$, it is evident that the equation can subsist only when $x=0$ and $y=0$; that is, when the ellipse is reduced to a point.

If in the same equation (V) A and B have contrary signs (which is the condition for an hyperbola), should $F=0$ there would result $y=\pm x\sqrt{\frac{B}{A}}$, which is an equation for two straight lines intersecting one another at the centre of the curve, where $x=0$. The positive and negative values of y

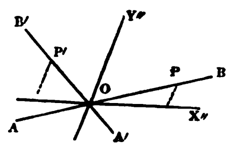
being equal to one another, it follo ws that the axis $O_{\prime}X_{\prime}$, on which x is measured, will bisect the angle formed by the intersecting lines.

When the equation (VI), which appertains to a parabola, becomes
$$Ay^2+Dy=F,$$
we have
$$y=-\frac{D}{2A}\pm\frac{1}{2A}\sqrt{(D^2+4AF)},$$
or the two values of y, if real, are constant; and the equation indicates merely two straight lines parallel to $O_{\prime}X_{\prime}$. Should F be negative, and, at the same time, $4AF$ be equal to D^2, y would have but one value $\left(=-\frac{D}{2A}\right)$, and the curve would be reduced to a single straight line. Again, F being negative, if $4AF$ exceed D^2, the value of y will be imaginary, and the equation leads to no result.

51. It has been seen (Art. 43.) that when the equation (A″) Art. 30. becomes
$$Ay^2+Bx^2+Dy+Ex=F,$$
that is, when $C=0$, the co-ordinate axes $O_{\prime}X_{\prime}$, $O_{\prime}Y_{\prime}$ are parallel to two conjugate diameters of the curve, and that their positions with respect to OX, OY (two co-ordinate axes making right or oblique angles with one another) are indicated by the equation
$$2A'v'v+2B'u'u+C'(u'v+v'u)=0:$$
now if OX, OY were parallel to some pair of conjugate diameters, either at right or oblique angles to one another, it is evident that the equation (A) Art. 29. would have a form corresponding to that which, in Art. 43., (A″) is assumed to have; that is, it would become
$$A'y'^2+B'x'^2+D'y'+E'x'=F',$$
or C′ would become zero. It follows that, if one pair of conjugate diameters be referred to another pair, as co-ordinate axes, the equation indicating their positions will be
$$A'vv'+B'uu'=0.$$

From this equation we have
$$\frac{v'}{u'}=-\frac{B'u}{A'v}, \text{ or } \frac{v}{u}=-\frac{B'u'}{A'v'},$$
the negative sign indicating that the angle MPD (Fig. to Art. 34.), or $\omega-\omega''$ $(=Y_{\prime}O_{\prime}Y'')$, Fig. to Art. 19.), is to be considered as a negative quantity; or that $O_{\prime}X''$ and $O_{\prime}Y_{\prime}$ are on opposite sides of $O_{\prime}Y''$.

If OX'', OY'' be two conjugate diameters making any angle with one another, and AB, $A'B'$ be another pair of conjugate diameters, the angle $X''OB$ is that which is designated ω'; $X''OB'$ or $X''OA'$, ω''; and $X''OY''$, ω.



Now let $x_{\prime}$ and $y_{\prime}$ be co-ordinates of any point P in AB, with respect to OX'', OY'' as axes; we shall then have (Art. 9.), on putting v' and u' for their equivalents (Art. 43.),

$$y_{\prime} = \frac{v'}{u'} x_{\prime}:$$

also if $x_{\prime\prime}$ and $y_{\prime\prime}$ be co-ordinates of any point P′ in A′B′ with respect to OX'', OY'' we have (Art. 9.), putting v and u for their equivalents (Art. 43.),

$$y_{\prime\prime} = \frac{v}{u} x_{\prime\prime}.$$

But $\frac{v}{u} = -\frac{B'u'}{A'v'}$; therefore the last equation becomes

$$y_{\prime\prime} = -\frac{B'u'}{A'v'} x_{\prime\prime}.$$

Consequently, representing $\frac{v'}{u'}$ by q, if the equation for any diameter be

$$y_{\prime} = q x_{\prime},$$

that of its conjugate diameter will be

$$y_{\prime\prime} = -\frac{B'}{A'q} x_{\prime\prime}.$$

52. A diameter is in general considered as a chord passing through the centre, and having its extremities in the curve line. In an ellipse this is always the case; but in a parabola all the diameters are of infinite length; and in an hyperbola two diameters which are conjugate to one another do not meet the same branch of the curve. Thus, let the co-ordinate axes be supposed to coincide, in direction, with any two conjugate diameters, when the equation of the curve will be (Art. 45.)

$$Ay^2 + Bx^2 = F, \text{ or } y^2 + \frac{B}{A} x^2 = \frac{F}{A}; \qquad (h)$$

then, since all the diameters pass through the origin of the co-ordinates, $y = qx$ may be the equation for a diameter; and, at the place where the curve is cut by the diameter, the values of y and x being the same both for the diameter

and the curve, this value of y may be substituted in the equation (h), which will then become

$$\frac{Aq^2+B}{A}x^2=\frac{F}{A};$$

whence $$x^2=\frac{F}{Aq^2+B}, \quad \text{or } x=\pm\frac{\sqrt{F}}{\sqrt{(Aq^2+B)}}.$$

In like manner, eliminating x^2, we have

$$y^2=\frac{Fq^2}{Aq^2+B}, \quad \text{or } y=\pm\frac{q\sqrt{F}}{\sqrt{(Aq^2+B)}}.$$

If A and B have like signs (in which case both of them are positive or may be made so), the values of x and y will always be real: but when A and B have like signs, the curve is an ellipse; therefore all the diameters of an ellipse cut the curve at their opposite extremities.

If A and B have unlike signs (which is the condition for an hyperbola), the denominators of both x and y become $\sqrt{(Aq^2-B)}$, and the values of those co-ordinates will be real only when Aq^2 is greater than B; in this case, therefore, a diameter will cut the curve.

But if $y=qx$ is the equation of any diameter, the equation of a diameter which is conjugate to it (the co-ordinate axes being coincident with any other pair of conjugate diameters) will be (Art. 51.)

$$y=-\frac{B}{Aq}x.$$

Substituting in (h) the value of y^2 obtained from this equation, we get, A and B having contrary signs,

$$x=\pm\frac{q\sqrt{AF}}{\sqrt{(B^2-BAq^2)}};$$

or, eliminating x,

$$y=\pm\frac{\sqrt{FB}}{\sqrt{(AB-A^2q^2)}}:$$

these values of x and y will become imaginary when Aq^2 is greater than B; whereas the co-ordinates of the diameter to which this is a conjugate have been shown to be real when $Aq^2>B$. Therefore a conjugate diameter of an hyperbola does not cut the same branch as is cut by the diameter to which it is a conjugate.

a. If, in the expressions for the co-ordinates of the diameters of an hyperbola, $Aq^2=B$, the values of x in the first equation and of y in the other become infinite; therefore a certain diameter of an hyperbola, and one which is conjugate

to it meet the curve at infinite distances from the origin of the co-ordinates.

b. DEF. 10. A line which meets a curve at an infinite distance from the origin of the co-ordinates, is called an asymptote.

c. The equation $\text{A}\,q^2=\text{B}$ relates to an asymptote; and from it we have $q^2=\frac{\text{B}}{\text{A}}$, or $q=\pm\sqrt{\frac{\text{B}}{\text{A}}}$. If, therefore, $x_{,}$ and $y_{,}$ be co-ordinates of any point in an asymptote, when referred to a pair of conjugate diameters at right, or at oblique angles to one other, we have

$$y_{,}=\pm\sqrt{\frac{\text{B}}{\text{A}}}\,x_{,}$$

for the equation of an asymptote so referred.

53. Now, let the equation (A'') Art. 30. become

$$\text{C}xy+\text{D}y+\text{E}x=\text{F}, \qquad (k)$$

in which case A and B, or the sums of the coefficients of y^2 and x^2 in (A') are separately zero.

From the equations thus formed we have

$$\text{C}'=-\frac{\text{A}'q^2+\text{B}'t^2}{qt} \text{ and } \text{C}'=-\frac{\text{A}'r^2+\text{B}'p^2}{pr}:$$

equating these values of C' we obtain

$$\text{A}'qr\,(pq-rt)=\text{B}'pt\,(pq-rt),$$

or

$$\text{A}'qr=\text{B}'pt. \qquad (l)$$

On squaring the first value of C' we have

$$\text{C}'^2=\text{A}'^2\frac{q^2}{t^2}+2\,\text{A}'\text{B}'+\text{B}'^2\frac{t^2}{q^2};$$

but, from (l) above,

$$\text{A}'^2\frac{q^2}{t^2}=\text{B}'^2\frac{p^2}{r^2}, \text{ and } 0=2\,\text{A}'\text{B}'-2\,\text{B}'^2\frac{pt}{qr};$$

therefore substituting, and adding the last equation to the result, we get

$$\text{C}'^2=4\,\text{A}'\text{B}'+\text{B}'^2\left(\frac{t^2}{q^2}-\frac{2pt}{qr}+\frac{p^2}{r^2}\right).$$

It follows that C'^2 is greater than $4\,\text{A}'\text{B}'$, and consequently that the curve is an hyperbola.

Substituting, in the above equation (l), the values of p, q, r, t (Art. 30.), omitting the denominator which is common to all the terms, that equation becomes

$$\text{A}'vv'=\text{B}'uu':$$

this (Art. 51.) is an equation for two conjugate diameters when referred to another pair of conjugate diameters; and to two such diameters the co-ordinate axes $\text{O}_{,}\text{X}_{,}$, $\text{O}_{,}\text{Y}_{,}$ are therefore parallel.

But the equation (k) may be put in the form

$$y+\frac{\text{D}y}{\text{C}x}+\frac{\text{E}}{\text{C}}=\frac{\text{F}}{\text{C}x},$$

or

$$x+\frac{\text{E}x}{\text{C}y}+\frac{\text{D}}{\text{C}}=\frac{\text{F}}{\text{C}y}.$$

The first, when x is infinite, gives $y=-\frac{\text{E}}{\text{C}}$; and the second, when y is infinite, gives $x=-\frac{\text{D}}{\text{C}}$. Therefore the co-ordinate axes are parallel to two conjugate diameters, which meet the curve at infinite distances from the origin of x and y: that is, they are parallel to two asymptotes.

a. When the equation (A'') is reduced to

$$\text{C}xy=\text{F},$$

the co-ordinate axes are evidently coincident with the asymptotes of the hyperbola.

54. If, in the general equation (A'') Art. 30., the coefficients A and B of y^2 and x^2 be equal to one another, that equation may be put in the form

$$y^2+x^2+\text{C}xy+\text{D}y+\text{E}x=\text{F}; \qquad (a)$$

and when the co-ordinate axes coincide with two conjugate diameters of the curve, so that $\text{C}=0$, $\text{D}=0$, and $\text{E}=0$, the equation becomes

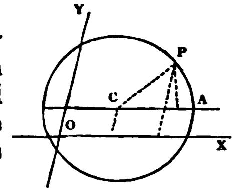
$$y^2+x^2=\text{F}:$$

this is manifestly the equation for a circle in which (x and y being, with respect to the centre, the rectangular co-ordinates of any point in its circumference) F represents the square of the semi-diameter.

Hence if, in the general equation (A'') Art. 30., and in the several transformed equations in Articles 41. 43. 44. and 45. we have $\text{A}=1$ and $\text{B}=1$, those equations will appertain to a circle: the first, when the co-ordinate axes have any position, and are inclined to one another at any angle, and the three last when the axes are at right angles to one another, and are

either parallel to or coincident with two diameters of the circle.

If OX, OY be co-ordinate axes in any position, making with one another an angle represented by φ, C the centre and P any point in the circumference of the circle; then α and β being the co-ordinates of the centre, we have



$$r^2\ (=\mathrm{CP}^2)=(x-\alpha)^2+(y-\beta)^2+2\,(x-\alpha)\,(y-\beta)\cos.\,\varphi,$$ (Pl. Trigon. Art. 57. *b.*)

or $y^2+x^2+2\cos.\,\varphi.\,xy-2\,(\beta+\alpha)\cos.\,\varphi.\,y-2\,(\beta+\alpha)\cos.\,\varphi.\,x$ $=r^2-(\alpha^2+\beta^2+2\,\alpha\beta\cos.\,\varphi)$:
corresponding to the equation (a) above.

a. In a curve of the second order each of two conjugate diameters is (Art. 38.) parallel to the chords which are bisected by the other; and (Euc. III. 3.) any diameter of a circle is perpendicular to the chords which it bisects: it follows that, in a circle, all the diameters which are at right angles to one another, are conjugate diameters.

55. The equations (A) and (A″) Arts. 29, 30., with those which have been assumed as modifications of them, having the same forms, whether the co-ordinate axes form right or oblique angles with one another; it follows that, when these axes are parallel to, or coincident with any pair of oblique conjugate diameters, the properties of the curves deduced from the relations between the co-ordinates of any point referred to those axes, will correspond precisely to the properties deduced from the relations between the co-ordinates of a point referred to axes which are parallel to, or coincident with the transverse and conjugate axes of the curve. Therefore, in the following propositions, no distinction of cases with respect to the oblique diameters and the axes is made, except when the subject relates in particular to these last.

CHAP. III.

PROPERTIES OF THE CONIC SECTIONS.

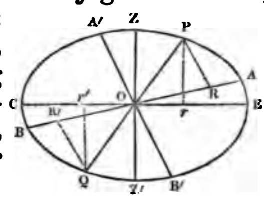
PROPOSITION I.

56. To find the equations for an ellipse in terms of any two semi-conjugate diameters; the origin being at the centre.

The equation for an ellipse, when a point in the curve is referred to two conjugate diameters, is (Art. 45.)

$$\mathrm{A}y^2+\mathrm{B}x^2=\mathrm{F}; \text{ whence } y^2=\frac{\mathrm{F}}{\mathrm{A}}-\frac{\mathrm{B}}{\mathrm{A}}x^2.$$

In the figure let AB, A′B′, be any two conjugate diameters, and O the origin of the co-ordinates: let OA, or OB, be represented by a, and OA′, or OB′, by b: also, P being any point in the curve, and the ordinate PR being drawn parallel to A′B′, let OR be represented by x, and RP by y.



Each of the co-efficients A and B, and the term F remaining the same, whatever values be assumed for either of the variables x or y; let, in the first place, $x=0$, and afterwards $y=0$: now, when $x=0$, we have

$$y^2\left(=\frac{\mathrm{F}}{\mathrm{A}}\right)=b^2,$$

and when $y=0$, $\quad x^2\left(=\frac{\mathrm{F}}{\mathrm{B}}\right)=a^2;$

hence $\quad \frac{\mathrm{B}}{\mathrm{A}}=\frac{b^2}{a^2}.$

Substituting these values in the equation for y^2, that equation becomes $\quad y^2=b^2-\frac{b^2}{a^2}x^2,$

which may be put in the form

$$y^2=\frac{b^2}{a^2}(a^2-x^2), \quad \text{or} \quad x^2=\frac{a^2}{b^2}(b^2-y^2). \qquad (a)$$

The same equation is also frequently put in the form

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}=1, \text{ or } a^2y^2+b^2x^2=a^2b^2. \qquad (b)$$

a. Since $y=0$ when x becomes either $+a$ or $-a$, and $x=0$ when y becomes $+b$, or $-b$; also, since y cannot become greater than b, nor x greater than a in the equations (a), without x in the one case, or y in the other, becoming imaginary, it is evident that the curve line returns into itself, and, when b is not equal to a, it has an oval form. It is evident also that for each equal value of x, one positive, an the other negative, there is a positive and a negative value of y equal to one another. Thus, OR′ being equal to OR, the ordinate R′Q, drawn parallel to A′B′, will be equal to PR; and OP, and OQ, being joined, the alternate angles at R and R′, P and Q will be equal to one another respectively; therefore, OQ=OP, the angle POR=QOR′, and OQ, OP, are in a straight line. Consequently, the line PQ, joining the opposite extremities of two equal ordinates, passes through O (where $x=0$, and $y=0$), and is a diameter of the curve.

The above equations hold good, whatever be the two conjugate diameters; and, if the transverse and conjugate axes CE and ZZ′ of the ellipse be taken for the axes of the coordinates, it is evident, being at right angles to one another, and the ordinates Pr, Qr', each at right angles to CE, that the axes will divide the curve into four parts, which are similar and equal to one another.

b. If A=B we have $b=a$, and the equation

$$y^2=\frac{b^2}{a^2}(a^2-x^2) \text{ [(}a\text{) above] becomes } y^2=a^2-x^2,$$

the equation of a circle of which a is the radius.

In this chapter, where it may be necessary to make the distinction, a and b will be put to represent any two conjugate semidiameters; t and c the semi-transverse, and semi-conjugate axes of a conic section.

PROPOSITION II.

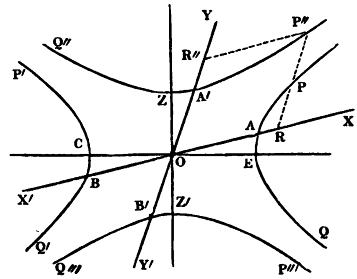
57. To find the equations for an hyperbola in terms of any two semi-conjugate diameters; the origin being at the centre.

The equation for an hyperbola, when a point in the curve

is referred to two conjugate diameters, is, (Arts. 45. and 48. *b. c.*), supposing A to be negative,

$-Ay^2+Bx^2=F$; whence

$$y^2=\frac{B}{A}x^2-\frac{F}{A}.$$



In the figure let AB, A′B′, be any two conjugate diameters, O the origin of the coordinates, and P any point in the curve, and let the ordinate PR be drawn parallel to A′B′: then the semi-diameters being represented as in Prop. I.,

when $\qquad x=0,\ y^2\left(=-\frac{F}{A}\right)=-b^2,$

and when $\qquad y=0,\ x^2\left(=\frac{F}{B}\right)=a^2;$

hence $\qquad \frac{B}{A}=\frac{b^2}{a^2}.$

Substituting these values in the equation for y^2 that equation becomes

$$y^2=\frac{b^2}{a^2}x^2-b^2,$$

which may be put in the form

$$y^2=\frac{b^2}{a^2}(x^2-a^2),\ \text{or}\ x^2=\frac{a^2}{b^2}(y^2+b^2): \qquad (a)$$

also in the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ \text{or}\ b^2x^2-a^2y^2=a^2b^2 \qquad (b)$$

Again, if B were made negative in the first equation; that is, if

$$Ay^2-Bx^2=F,\ \text{or}\ x^2-\frac{A}{B}y^2=-\frac{F}{B};$$

we should, by proceeding in like manner, obtain

$$x^2=\frac{a^2}{b^2}(y^2-b^2)\ \text{or}\ \frac{y^2}{b^2}-\frac{x^2}{a^2}=1,\ \text{or}\ a^2y^2-b^2x^2=a^2b^2, \qquad (b')$$

or again $\qquad y^2=\frac{b^2}{a^2}(x^2+a^2). \qquad (c)$

a. From the first of the equations (*a*) it is manifest that, while x has any value between zero and a; that is, while x, either positive or negative, is less than OA or OB, the value

of y, an ordinate to A B, is imaginary. The value of y is real when x is greater than a, and it becomes infinite when x is infinite: also, for every value of x, both positive and negative, there are two equal values of y, the one positive and the other negative; and hence the curve has infinite branches, P E Q, P′C Q′, extending from A towards X, and from B towards X′.

The transverse and conjugate axes C E, Z Z′, of the hyperbola, being at right angles to one another, it is evident that the branches E P, E Q, C P′, C Q′ are similar and equal to one another.

b. What has been said of the curves P E Q, P′C Q′ is true also of the curves P″Z Q″, P‴Z′Q‴: these are designated by the equations (b'), in which it is evident that x, an ordinate to A′B′, is imaginary, while y is less than b, that is, less than O A′ or O B′.

Each of the curves last mentioned is called a conjugate hyperbola with respect to each of the parts P E Q, P′C Q′. If the transverse and conjugate axes C E, Z Z′ of an hyperbola are equal to one another, the first equation (a) will become (t representing O E, and, in this case, O Z also)

$$y^2=x^2-t^2.$$

Such an hyperbola is said to be equilateral.

The equation $y^2=\frac{b^2}{a^2}(x^2-a^2)$, which appertains to either of the curves P E Q, P′C Q′, may be transformed into the equation $x^2=\frac{a^2}{b^2}(y^2-b^2)$, which appertains to either of the branches P″Z Q″, P‴Z′Q‴, of the conjugate hyperbola on putting x^2 for y^2 and y^2 for x^2, also a^2 for b^2 and b^2 for a^2, the co-ordinates x and y being on the same axes O X, O Y respectively in both cases; but it is necessary to observe that, in the first case, y represents P R; in the other, and also in the equation $y^2=\frac{b^2}{a^2}(x^2+a^2)$, it represents R P″ or O R″.

If O R″ be represented by x, and R″P″ by y, the first equation (a) may be transformed into the corresponding equation for the conjugate hyperbola by simply putting a for b and b for a; when we shall obtain

$$y^2=\frac{a^2}{b^2}(x^2-b^2).$$

c. Since, for an hyperbola, A and B have contrary signs,

and that $\frac{B}{A} = \frac{b^2}{a^2}$, in which a and b are any conjugate diameters; it is evident that if a^2 represent the square of any semi-diameter, the square of the conjugate semi-diameter must be represented by $-b^2$, or the conjugate diameter itself by $\sqrt{-b^2}$ or by $b\sqrt{-1}$: and thus the conjugate diameters in an hyperbola are considered as imaginary.

58. From the equation (IV) Art. 44. $a.$, or its equivalent

$$y^2 + \frac{B}{A}x^2 + \frac{E}{A}x = \frac{F}{A},$$

there may be found, in terms of two conjugate semi-diameters, an equation for an ellipse or hyperbola when the origin of the co-ordinates is at one extremity B of a diameter BA; one of the co-ordinate axes being co-incident in direction with that diameter, and the other parallel to the conjugate diameter; consequently (Art. 38.) a tangent to the curve.

In this case, when $x=0$, we have $y=0$, and therefore $\frac{F}{A}=0$ (as in Art. 49.); also when $x=a$, $y=b$; and when $x=2a$, $y=0$.

Substituting these two last values of x and y successively in the equation, and reducing, we get (A and B having like signs) for the ellipse,

$$y^2 = \frac{b^2}{a^2}(2ax - x^2)^* \qquad (a)$$

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* Thus: Substituting in the equation (IV) a for x, and b for y, that equation becomes, $\frac{F}{A}$ being zero,

$$b^2 + \frac{B}{A}a^2 + \frac{E}{A}a = 0$$

Substituting in the same equation $2a$ for x, and 0 for y, it becomes

$$\frac{B}{A}4a^2 + \frac{E}{A}2a = 0.$$

Doubling the first of these equations and subtracting,

$$2b^2 - \frac{B}{A}2a^2 = 0;\ \text{whence}\ \frac{B}{A} = \frac{b^2}{a^2}:$$

also, quadrupling the first and subtracting,

$$4b^2 + \frac{E}{A}2a = 0;\ \text{whence}\ \frac{E}{A} = -\frac{2b^2}{a}.$$

Substituting these values of $\frac{B}{A}$ and $\frac{E}{A}$, and observing that $\frac{F}{A}=0$, the ori-

The corresponding equation for the circle is (a being the semi-diameter)

$$y^2=2ax-x^2.$$

If A and B have unlike signs we may obtain from (IV) in a similar manner, for the hyperbola (see the note),

$$y^2=\frac{b^2}{a^2}(2ax+x^2). \qquad (b)$$

But these equations for the ellipse and hyperbola may be obtained more readily on substituting $a-x$ for x in the equation

$$y^2=\frac{b^2}{a^2}(a^2-x^2),$$

and $a+x$ for x in the equation

$$y^2=\frac{b^2}{a^2}(x^2-a^2).$$

59. Since, in the ellipse and hyperbola $\frac{\mathrm{B}}{\mathrm{A}}$ represents $\frac{b^2}{a^2}$, it follows that the equation

$$y_{\prime}=\pm x_{\prime}\sqrt{\frac{\mathrm{B}}{\mathrm{A}}} \qquad \text{(Art. 52. } c.)$$

of an asymptote referred to two conjugate diameters becomes, omitting the sub-accents,

$$y=\pm\frac{b}{a}x, \text{ or } x=\pm\frac{a}{b}y,$$

which, when the conjugate semi-diameters are the major and minor axes of the curve, is

$$y=\pm\frac{c}{t}x, \text{ or } x=\pm\frac{t}{c}y.$$

60. COR. The double sign prefixed to the values of y indi-

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ginal equation becomes

$$y^2+\frac{b^2}{a^2}x^2-\frac{2b^2a}{a^2}x \quad ;$$

whence, for the ellipse,

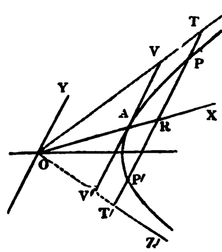
$$y^2=\frac{b^2}{a^2}(2ax-x^2).$$

In a similar manner, for the hyperbola, making the above value of $\frac{\mathrm{B}}{\mathrm{A}}$ negative (A and B having unlike signs), may be obtained the equation

$$y^2=\frac{b^2}{a^2}(2ax+x^2).$$

cates that for every value of x there are two equal values of that ordinate.

Thus, let OX, OY be co-ordinate axes coincident with two conjugate diameters of an hyperbola; then, for every value of x (= OR) there will be two equal values (RT and RT′) of y, the points T and T′ being in the asymptotes OZ, OZ′. But for every value of x there are (Art. 57. *a.*) two equal values of the corresponding ordinates (RP, RP′) of the hyperbola; therefore, taking equals from equals, PT = P′T′. Thus when any chord in an hyperbola is produced to meet the asymptotes, the segments between the curve and the asymptotes are equal to one another. It follows also, VAV′ being a tangent at A, that AV = AV′ and that each is equal to b.



PROPOSITION III.

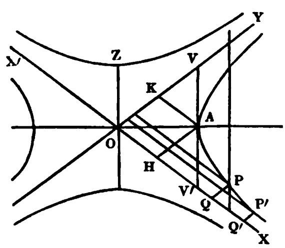
61. To find the equation for an hyperbola with respect to the asymptotes in terms of the transverse and conjugate axes.

From Art. 53. *a.* we have, when the co-ordinate axes are coincident with the asymptotes OX, OY of an hyperbola,

$$Cxy = F,$$

which may be put in the form

$$xy = F.'$$



Now let OA be the semi-transverse axis of the hyperbola and OZ, at right angles to it the semi-conjugate axis; and let the asymptote OY be referred to these axes; then the equation of the asymptote will be

$$y_{,} = \pm \frac{c}{t} x_{,};$$

consequently, when $x_{,} = t$ (= OA), $y_{,} = \pm c$ (= AV or AV′ drawn through A at right angles to OA, to meet OY and OX). Now, in the triangles VOA, AOV′, AV = AV′, OA is common, and the angles OAV, OAV′ are right angles; therefore the angle VOV′ between the asymptotes is bisected by OA; also YOX′ is bisected by OZ. Next, AK being drawn parallel to OX and AH to OY, it may be proved that OK, KV, KA, OH, HV′, HA are equal to one another; for the angle VOV′ being bisected by

OA and the alternate angles at A being equal; the angles AOK, OAK (=AOH) are equal to one another, and consequently OK=KA; in like manner OH=HA, and the triangles OKA, OHA being equal, OK=OH, &c.; also OAV being a right angle, VAK and AVK are each equal to the complement of AOK; therefore KV=KA, and similarly HV′=HO=HA.

But OV^2 ($=OA^2+AV^2$) $=t^2+c^2$, and $OK^2=\frac{1}{4}OV^2$; therefore OK^2, OH^2 and HA^2 are each equal to $\frac{1}{4}(t^2+c^2)$. Now OH and HA are particular values of x and y, the co-ordinates of the curve referred to the asymptotes; therefore, xy (=F′) being constant,

$$xy=\tfrac{1}{4}(t^2+c^2).$$

If the second member be represented by m^2, we have

$$xy=m^2.$$

62. COR. If any number of parallelograms, as OP, OP′, be formed by drawing lines, as PQ, P′Q′, parallel to the asymptote OY, those parallelograms will manifestly be equivalent to one another; for the angle YOX being common to all, let it be represented by ω; then OH and HA being each represented by m, OQ or OQ′ by x, and QP or Q′P′ by y, on multiplying both members of the above equation by sin. ω we have

$$xy \text{ sin. } \omega=m^2 \text{ sin. } \omega:$$

and (Pl. Trigon. Art. 75. a.) these members denote, respectively, the areas of the parallelograms OP, or OP′ and KH: thus the equivalence of the parallelograms is proved.

a. Since x and y represent the co-ordinates of any point, as P, P′, &c., it is evident, from the equation $xy=m^2$, that among the co-ordinates of points in the hyperbola, with respect to the asymptotes, there exist the following proportions:—

OH : OQ :: PQ : OK, OH : OQ′ :: P′Q′ : OK, &c.;

whence

OH . OK=OQ . PQ=OQ′ . P′Q′, &c.,

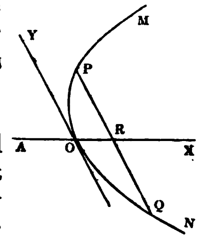
or OH, OQ, OQ′, &c. are reciprocally as OK, PQ, P′Q′, &c.

PROPOSITION IV.

63. To find the equation for a parabola in terms of an abscissa or segment of any diameter and the corresponding ordinate.

The equation for a parabola, when one of the co-ordinate axes coincides with a diameter and the other is parallel to

one of the ordinates of that diameter, or coincides with a tangent to the curve at the extremity of the diameter (the origin of the co-ordinates being then at that extremity of the diameter), is (Arts. 46. *a.* and 49.),



$$\text{A}y^2+\text{E}x=0.$$

Let OX, OY be the co-ordinate axes, and MON a portion of the parabolic curve. Let any segment OR, of OX, be represented by a; RP, the corresponding ordinate, by b; that is, when $x=a$, let $y=b$. Then the equation becomes, (E being considered as negative),

$$\text{A}b^2=\text{E}a; \text{ whence } \frac{\text{E}}{\text{A}}=\frac{b^2}{a}.$$

Therefore, by substitution,

$$y^2=\frac{b^2}{a}x. \qquad (a)$$

If the origin of the co-ordinates were at A in XO produced, an equation for the parabola, in the like terms, might be obtained from the equation

$$\text{A}y^2+\text{E}x=\text{F};$$

or, more simply (representing AO by m, and AR being x'), by substituting $x'-m$ for x in the equation above for y^2; which gives, a and b having the same values as before,

$$y^2=\frac{b^2}{a}(x'-m).$$

64. The equations for an ellipse, an hyperbola, and a parabola, may be converted into proportions, in which form the relations between the co-ordinates x and y are frequently stated. Thus,

From (*a*) Art. 56. we have	$a^2 : b^2 :: a^2-x^2) : y^2$,
or	$a^2 : b^2 :: (a+x)(a-x) : y^2$,
or again, from (*a*), Art. 58.	$a^2 : b^2 :: 2ax-x^2 : y^2$.
From (*a*) art. 57. we have	$a^2 : b^2 :: x^2-a^2 : y^2$,
or	$a^2 : b^2 :: (x+a)(x-a) : y^2$,
or again, from (*b*), Art. 58.	$a^2 : b^2 :: 2ax+x^2 : y^2$:
and from (*a*) Art. 63. we have	$a : b^2 :: x : y^2$.

In the six first proportions, or those which appertain to the ellipse and hyperbola, the terms a^2-x^2 and $2ax-x^2$, x^2-a^2 and $2ax+x^2$ denote the products of the abscissæ or segments, between the place of the ordinate y and the two extremities of the diameter. Of these curves, therefore, it

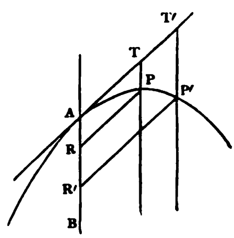
may be stated that *the products of the abscissæ of any diameter, are to one another as the squares of the corresponding ordinates.* The last proportion, or that which appertains to a parabola, indicates that *the abscissæ of any diameter are to one another as the squares of the corresponding ordinates.*

a. COR. to the proposition. If a tangent at the extremity of a diameter, as AB of a parabola, be cut in any points T and T′ by diameters drawn through points, as P and P′ in the curve, we shall have

$$\text{PT} : \text{P}'\text{T}' :: \text{AT}^2 : \text{AT}'^2.$$

For drawing ordinates PR, P′R′ to the diameter AB, these will be parallel to AT′, and the diameters being parallel to one another, the figures RT, R′T′ will be parallelograms; hence AR, AR′ will be respectively equal to TP, T′P′ and RP, R′P′ to AT, AT′. Now (as above) $\text{AR} : \text{AR}' :: \text{RP}^2 : \text{R}'\text{P}'^2$; therefore, substituting equals for equals,

$$\text{PT} : \text{P}'\text{T}' :: \text{AT}^2 : \text{AT}'^2.$$



65. DEF. 11. A third proportional to any diameter and its conjugate, is called the parameter of that diameter.

Thus, in an ellipse or hyperbola, a being any semi-diameter and b one which is conjugate to it; also, p representing the parameter of the former diameter,

$$2a : 2b :: 2b : p;$$

consequently $$p = \frac{2b^2}{a}.$$

In a parabola, a, on any diameter, being an abscissa measured from the vertex or intersection of the diameter with the curve line, and b the corresponding ordinate, the proportion is

$$a : b :: b : p;$$

therefore the parameter of any diameter of a parabola is expressed by $\frac{b^2}{a}$.

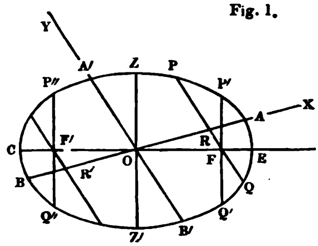
The parameter of the transverse axis of an ellipse or hyperbola is sometimes called the *latus rectum.* Also a constant line appertaining to any curve, or a term in the equation for any curve, which, on being made to vary, adapts the equation to other curves of a like kind, is called a parameter.

PROPOSITION V.

66. To find a point in any diameter of an ellipse, an hyperbola, or a parabola, through which a chord being drawn parallel to the conjugate diameter, such chord shall be equal to the parameter of the former diameter.

Let AB, A′B′ be any two conjugate diameters of an ellipse, with which the co-ordinate axes OX, OY coincide, and let PQ, equal to the parameter of AB, be a chord parallel to A′B′; it is required to find in AB the point R, in which PQ cuts that diameter.

Fig. 1.



Let OA $=a$, OA′ $=b$, and OR $=x$; then RP, or y, (equal to half the parameter of AB) $=\frac{b^2}{a}$.

By the equation for the ellipse, (a) Art. 56., we have

$$\frac{b^2}{a^2}(a^2-x^2)=\frac{b^4}{a^2};$$

hence $a^2-x^2=b^2$, or $x^2=a^2-b^2$, or $x=\pm\sqrt{(a^2-b^2)}$.

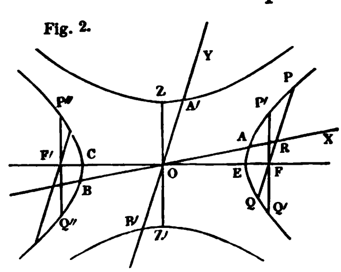
The double sign indicates that there are two equal values of x, OR and OR′, through which chords being drawn parallel to A′B′, each of them will be equal to the parameter of AB.

When the diameter and its conjugate coincide with the transverse and conjugate axes EC, ZZ′ of the ellipse, t and c representing the semi-axes, we have

$$x^2=t^2-c^2, \text{ or } x=\pm\sqrt{(t^2-c^2)}. \qquad (m.)$$

The points F and F′ on CE, at distances from O equal to this value of x, are points through which chords P′Q′, P″Q″, drawn perpendicular to CE, are each equal to the parameter of the axis EC.

Fig. 2.



For an hyperbola, the co-ordinate axes OX and OY coinciding with any two conjugate diameters, on using the equation

$$\frac{b^2}{a^2}(x^2-a^2)=y^2, \qquad \text{[(Art. 57.]}$$

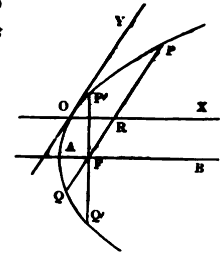
and substituting for y^2 $(=\text{RP}^2)$ its equivalent $\frac{b^4}{a^2}$ (the square of half the parameter of AB), we have

$$\frac{b^2}{a^2}(x^2-a^2)=\frac{b^4}{a^2};$$

hence $x^2-a^2=b^2$, or $x^2\ (=\text{OR}^2)=a^2+b^2$.

When the conjugate diameters coincide with the axes CE, ZZ′ of the hyperbola, $x^2=t^2+c^2$, or $x=\pm\sqrt{(t^2+c^2)}$: (n) the two values of x (OF and OF′) are on the transverse axes produced; and if at the points F and F′ chords be drawn perpendicular to CE, each of those chords is equal to the parameter of the axis.

For a parabola, the co-ordinate axes OX, OY coinciding with any diameter and with a tangent to the curve at the extremity; on using the equation



$$\frac{b^2}{a}x=y^2, \qquad \text{[(a) Art. 63.]}$$

and substituting for y^2 $(=\text{RP}^2)$ its equivalent $\frac{b^4}{4a^2}$ the square of half the parameter of OX (Art. 65.), we have

$$\frac{b^2}{a}x=\frac{b^4}{4a^2},\ \text{or}\ x\ (=\text{OR})=\frac{b^2}{4a}. \qquad (p)$$

When the diameter OX coincides with the axis AB of the curve, a like expression will be found for x $(=\text{AF})$; and at the point F the chord P′Q′, at right angles to AB, will be equal to the parameter of the axis.

67. DEF. 12. A point, as F, or F′, on the transverse axis of an ellipse or hyperbola, and on the axis of a parabola, through which a chord being drawn perpendicularly to the axis is equal to the latus rectum, or parameter of the axis, is called a focus.

It is manifest that an ellipse or hyperbola has on its transverse axis two foci, which are at equal distances from the centre: a parabola has but one focus.

68. DEF. 13. The distance (OF, or OF′) of a focus from the centre of an ellipse or hyperbola is called the excentricity of the curve. The excentricity is generally represented by e.

69. From the equation (m) Art. 66. we have

$$e^2\ (=t^2-c^2)=(t+c)\,(t-c),$$

or the excentricity of an ellipse is a mean proportional between the sum and difference of the semi-axes.

70. In an ellipse, also, since (Fig. 1. to Art. 66.)

$$\text{FZ}^2\ (=\text{OF}^2+\text{OZ}^2)=e^2+c^2,$$

and from this proposition $t^2=e^2+c^2$, it is evident that

$$\text{FZ}^2=t^2, \quad \text{or} \quad \text{FZ}=\pm t;$$

or the distance of either focus from either extremity of the conjugate axis is equal to the semi-transverse axis.

71. From the equation (n) we have $e^2=t^2+c^2$; and since (Fig. 2. to Art. 66.) $\text{EZ}^2\ (=\text{OE}^2+\text{OZ}^2)=t^2+c^2$, it is evident that

$$\text{EZ}^2=e^2, \quad \text{or} \quad \text{EZ}=\pm e;$$

or the length of a line joining either extremity of the transverse to either extremity of the conjugate axis of an hyperbola, is equal to the excentricity.

72. From the equation (p) it is evident that, on any diameter of a parabola, the distance from the extremity of that diameter to the point through which is drawn a chord parallel to the ordinates, and equal to the parameter of that diameter, is one-fourth of that parameter: and the distance from the focus to the extremity of the axis is one-fourth of the parameter of the axis.

If p represent the parameter of any diameter of a parabola, the equation (a) Art. 63. may be put in the form

$$y^2=px.$$

Scholium. An ellipse, an hyperbola, or a parabola being given, the axes and foci may be found geometrically, thus:

1. For an ellipse: draw any two chords, as ab, cd, (Fig. 1.) parallel to one another; and having bisected them in p and q, draw through those points a straight line to cut the curve in opposite points, as P and Q: this line will (Art. 35.) be a diameter; and, bisecting it in O, the point O will be the centre of the curve. With O as a centre, and OP as a radius,

Fig. 1.

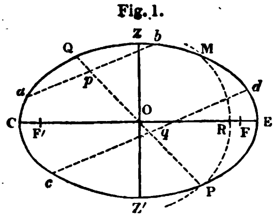
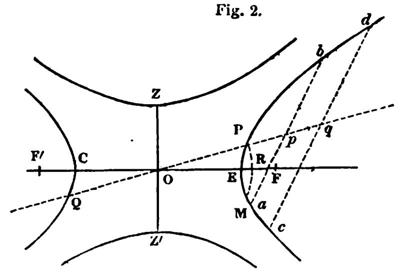


Fig. 2.

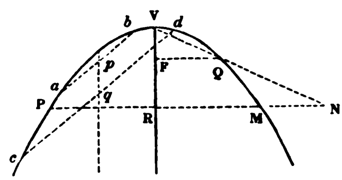


describe the circular arc PM: bisect this arc in R, and through

OR draw the diameter CE: this will be perpendicular to a chord, or double ordinate, joining P and M; therefore it will be one of the axes, and if the circular arc PM fall within the ellipse, as in the figure, it will be the transverse axis. The diameter ZOZ′, at right angles to CE, will be the other axis (the conjugate axis in the figure): and if from Z as a centre, with a radius equal to OC or OE, arcs be described cutting the transverse axis at F and F′, these points will (Art. 70.) be the foci.

2. By a process exactly similar to that which has been described may the axes CE, ZZ′ (Fig. 2.) of an hyperbola be found; and making OF, OF′ each equal to EZ, F and F′ will (Art. 71.) be the foci.

3. For a parabola: draw any two chords ab, cd parallel to one another; bisect them in p and q, and draw a line through those points. This line will (Art. 35.) be a diameter. Perpendicularly to pq draw a chord as PM, and produce it; bisect PM in R, and through R draw a line, as VR, perpendicularly to PM: this will consequently be the axis. Make RN equal to twice VR, and join V, N, VN will cut the curve in some point as Q: draw a line through Q parallel to PN, cutting the axis VR in F; then, by the similarity of the triangles VRN, VFQ, and since RN $=$ 2VR, the line FQ$=$2VF: consequently (Art. 72.) the double ordinate at F is the parameter of the axis, and F is the focus of the curve.



PROPOSITION VI.

73. The equation for a diameter of an ellipse or hyperbola being given, to find the equation for the conjugate diameter. And the equation for a chord of a parabola being given, to find the equation for the diameter which bisects it.

It has been shown (Art. 51.), that if the equation for any diameter of a conic section be represented by $y_{\prime}=qx_{\prime}$, the equation for a diameter conjugate to it, when referred to another pair of conjugate diameters, as axes of the co-ordinates, will be

$$y_{\prime\prime}=-\frac{\text{B}}{\text{A}q}\,x_{\prime\prime}.$$

Now, in the ellipse, $\frac{B}{A} = \frac{b^2}{a^2}$; or, when the conjugate diameters to which the first pair is referred are the axes of the curve, $\frac{B}{A} = \frac{c^2}{t^2}$; therefore the equation for a diameter of an ellipse being

$$y_{,}=qx_{,},$$

the equation of the conjugate diameter will be, with respect to any other pair of conjugate diameters,

$$y_{,,}= -\frac{b^2}{a^2q}x_{,,}, \text{ or with respect to the axes, } y_{,,}= -\frac{c^2}{t^2q}x_{,,}. \qquad (a)$$

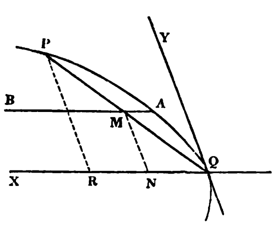
In the hyperbola, B and A having contrary signs, we have

$$\frac{B}{A} = -\frac{b^2}{a^2};$$

therefore, the equation for a diameter being $y_{,}=qx_{,}$, the equation for the conjugate diameter will be, with respect to any other pair of conjugate diameters,

$$y_{,,}=\frac{b^2}{a^2q}x_{,,}, \text{ or with respect to the axes, } y_{,,}=\frac{c^2}{t^2q}x_{,,}. \qquad (b)$$

In a parabola, the equation for a chord being given, the equation for the diameter which bisects it may be found in the following manner.



Let PQ be any chord, and let QX, QY be co-ordinate axes, the former being a diameter, and the other coincident with a tangent to the curve at Q, or parallel to the chords which are bisected by QX: also let AB be a diameter bisecting PQ in M.

The equation of the chord may be represented by $y=qx$;
and, p representing the parameter of the diameter QX, we have (Art. 72.) $y^2(=,$ for the point P, $\text{PR}^2)=px$.
From the equation of the chord we have, for the point P, $x=\frac{y}{q}$; therefore substituting,

$$y^2 = \frac{py}{q}, \text{ or } y=\frac{p}{q}.$$

But the ordinate (MN) of AB, with respect to QX, is constant and equal to the half of PR; therefore, representing that ordinate by y', we have, for the equation of the diameter AB,

$$y'\left(=\frac{y}{2}\right)=\frac{p}{2q}. \qquad (c)$$

Hence, q being equal to $\frac{p}{2y'}$, if the equation $y_{,}=qx_{,}$, of a diameter were given, this value of q might be substituted in it, and we should have

$$y_{,}=\frac{p}{2y'}\,x_{,}, \text{ or } y_{,}=\frac{b^2}{2ay'}\,x_{,} \qquad (c')$$

for the equation of the chord which is bisected by the diameter.

74. COR. In the ellipse and hyperbola, q the co-efficient of $x_{,}$, in the equation of a diameter, represents $\frac{v'}{u'}$, the ratio of the sines of the angles which the diameter makes with the two co-ordinate axes (Art. 43.), or it may represent the ratio of the co-ordinates of any point in that diameter; and, when the co-ordinate axes are the transverse and conjugate axes of the curve, the angles just mentioned are complements of one another, in which case q becomes the tangent of the angle which the diameter makes with the transverse axis. Let this angle be represented by θ: then the equation of the diameter being

$$y_{,}=x_{,}\tan.\theta,$$

the equation of its conjugate diameter will be, (a) Art. 73., for the ellipse,

$$y_{,,}=-x_{,,}\frac{c^2}{t^2\tan.\theta}.$$

Now $\frac{-c^2}{t^2\tan.\theta}$ represents the tangent of the angle which the conjugate diameter makes with the transverse axis of the curve: let this angle be represented by θ'. Then, for two conjugate diameters referred to the axes of the ellipse,

$$\frac{c^2}{t^2\tan.\theta}=-\tan.\theta', \text{ or } \frac{c^2}{t^2}=-\tan.\theta\tan.\theta',$$

or, again $\quad t^2\sin.\theta\sin.\theta'=-c^2\cos.\theta\cos.\theta'.$

For the hyperbola we have

$$t^2\sin.\theta\sin.\theta'=c^2\cos.\theta\cos.\theta'.$$

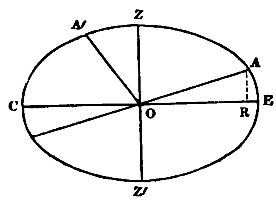
The first member of each of these equations is equivalent to the rectangle or product of two perpendiculars let fall from one extremity of the transverse axis of an ellipse or hyperbola on two conjugate diameters which make with that

axis angles equal to θ and θ', and the second member of each is equivalent to the rectangle or product of two perpendiculars let fall from one extremity of the conjugate axis upon the same conjugate diameters. It follows that the two rectangles are equivalent to one another.

PROPOSITION VII.

75. To find the length of any semi-diameter of an ellipse or hyperbola in terms of rectangular co-ordinates.

Let CE, ZZ′ be the transverse and conjugate axes of an ellipse, O its centre, and let OA be any semi-diameter. The co-ordinate axes being supposed to coincide with the axes of the curve, AR drawn perpendicularly to OE, will be an ordinate to the transverse axis.



Let OE be represented by t, OZ by c and OA by a; then, (a) Art. 56.

$$y^2\ (=\text{AR}^2)=\frac{c^2}{t^2}(t^2-x^2)$$

adding x^2 ($=\text{OR}^2$) to the second member, we have

$$a^2\ (=\text{OA}^2)=\frac{t^2c^2-c^2x^2+t^2x^2}{t^2}=\frac{t^2c^2+(t^2-c^2)\,x^2}{t^2}; \qquad (a)$$

or introducing the excentricity, by putting t^2-e^2 for c^2, and e^2 for t^2-c^2 (Art. 69), we get

$$a^2=\frac{t^2\,(t^2-e^2)+e^2x^2}{t^2}.$$

Or, when $t=1$, putting e' ($=\frac{e}{t}$) for the excentricity,

$$a^2=1-e'^2+e'^2x^2.$$

For the hyperbola, in which $y^2=\frac{c^2}{t^2}(x^2-t^2)$, ($a$) Art. 57. we get

$$a^2=\frac{(t^2+c^2)\,x^2-t^2c^2}{t^2}=\frac{e^2x^2-t^2\,(e^2-t^2)}{t^2}; \qquad (b)$$

or when $t=1$,

$$a^2=1-e'^2+e'^2x^2.$$

PROPOSITION VIII.

76. To find the length of any semi-diameter of an ellipse or hyperbola in terms of polar co-ordinates.

Let the angle EOA, in the above figure, be represented by θ; then a representing OA, and x being put for OR, since $a \cos. \theta = x$; on substituting $a \cos. \theta$ for x in the equation (a) Art. 75. the latter becomes

$$a^2 = \frac{t^2 c^2 + (t^2 - c^2) a^2 \cos.^2 \theta}{t^2};$$

whence $\quad t^2 a^2 = t^2 c^2 + (t^2 - c^2) a^2 \cos.^2 \theta,$

or $\quad t^2 a^2 - (t^2 - c^2) a^2 \cos.^2 \theta = t^2 c^2;$

consequently,

$$a^2 = \frac{t^2 c^2}{t^2 - (t^2 - c^2) \cos.^2 \theta}. \qquad (a)$$

Or, introducing the excentricity in the denominator by putting e^2 for $t^2 - c^2$; we have

$$a^2 = \frac{t^2 c^2}{t^2 - e^2 \cos.^2 \theta}; \qquad (a')$$

which, when $t = 1$, putting then e' for the excentricity, becomes, since, from Art. 69. $t^2 - e^2 = c^2$,

$$a^2 = \frac{1 - e'^2}{1 - e'^2 \cos.^2 \theta}.$$

For the hyperbola, putting $a \cos. \theta$ for x in the first of the equations (b) Art. 75. the latter becomes, after reduction,

$$a^2 = \frac{t^2 c^2}{(t^2 + c^2) \cos.^2 \theta - t^2}; \qquad (b)$$

or, introducing the excentricity in the denominator,

$$a^2 = \frac{t^2 c^2}{e^2 \cos.^2 \theta - t^2};$$

which, when $t = 1$, putting e' for the excentricity, becomes

$$\frac{e'^2 - 1}{e'^2 \cos.^2 \theta - 1}.$$

The value of a in the hyperbola becomes evidently infinite when $e^2 \cos.^2 \theta = t^2$, or $e'^2 \cos.^2 \theta = 1$.

The equations (b) Art. 75. and (b) above may be derived from (a) Art. 75. and (a) above, respectively, by merely changing the signs of c^2.

The equation (a) in this proposition may be again transformed as follows:—

$$a^2 = \frac{t^2\, c^2}{t^2 - t^2 \cos.^2 \theta + c^2 \cos.^2 \theta}, \text{ or} = \frac{t^2\, c^2}{t^2 \sin.^2 \theta + c^2 \cos.^2 \theta}; \quad (c)$$

and (b) in this proposition may be transformed into

$$a^2 = \frac{t^2\, c^2}{c^2 \cos.^2 \theta - (t^2 - t^2) \cos.^2 \theta}, \text{ or } a^2 = \frac{t^2\, c^2}{c^2 \cos.^2 \theta - t^2 \sin.^2 \theta}. \quad (c')$$

a. The length of any semi-diameter as A′O of an ellipse in terms of its conjugate semi-diameter AO may be easily found, thus:

Putting θ' for θ in (a), we have in an ellipse

$$b^2 \; (= \text{A}'\text{O}^2) = \frac{t^2\, c^2}{t^2 - (t^2 - c^2) \cos.^2 \theta'}$$

in which for $\cos.^2 \theta'$, putting its equivalent $\frac{1}{1 + \tan.^2 \theta'}$ (Pl. Trigon. Art. 24. *e.*) and for $\tan.^2 \theta'$ its equivalent $\frac{c^4}{t^4 \tan.^2 \theta}$ (Art. 74.) there will be obtained, after some reductions,

$$b^2 = \frac{t^4 \sin.^2 \theta + c^4 \cos.^2 \theta}{t^2 \sin.^2 \theta + c^2 \cos.^2 \theta}.$$

Comparing this with the value of a^2 (c) we find

$$b^2 = a^2 \frac{t^4 \sin.^2 \theta + c^4 \cos.^2 \theta}{t^2\, c^2}.$$

PROPOSITION IX.

77. If ordinates be drawn from the extremities of two conjugate diameters to any other diameter of an ellipse or hyperbola; in an ellipse the sum of the squares of the abscissæ from the centre of the curve, and in an hyperbola the difference of the squares of the abscissæ, will be equal to the square of half the diameter on which the ordinates fall.

Let AB, A′B′ be any two conjugate diameters, and let HK, MN be any other pair of conjugate diameters; also let AR, B′R′ or A′R″ be ordinates on HK: then, in an ellipse,

$$\text{OR}^2 + \text{OR}'^2, \text{ or } \text{OR}^2 + \text{OR}''^2 = \text{OH}^2.$$

Let OH $= a$, OM $= b$, OR $= x$, OR$' = x'$, AR $= y$ and R$'$B$' = y'$.

Fig. 1.

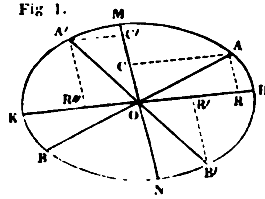
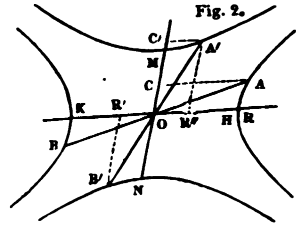


Fig. 2.



Let the equation for AB, when referred to KH and MN as axes, be $qx = y$; then the equation for A$'$B$'$, when referred to the same axes, will be (Art. 73.)

$$\mp \frac{b^2}{a^2 q} x' = y'.$$

Now, in an ellipse,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \text{ and } y'^2 = \frac{b^2}{a^2}(a^2 - x'^2) \text{ [(a) Art. 56.]};$$

therefore, for the point A, $q^2 x^2 = \frac{b^2}{a^2}(a^2 - x^2)$,

and for the point A$'$ or B$'$, $\frac{b^4}{a^4 q^2} x'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$.

From the first of these equations we have

$$q^2 = \frac{b^2}{a^2} \cdot \frac{a^2 - x^2}{x^2},$$

and from the second,

$$q^2 = \frac{b^2}{a^2} \cdot \frac{x'^2}{a^2 - x'^2}.$$

Equating, $$\frac{a^2 - x^2}{x^2} = \frac{x'^2}{a^2 - x'^2}$$

or $$a^4 - a^2 x^2 - a^2 x'^2 + x^2 x'^2 = x^2 x'^2;$$

whence $a^2 = x^2 + x'^2$, or (Fig. 1.) OH$^2 =$ OR$^2 +$ OR$'^2$.

It may be shown in like manner that OM$^2 =$ OC$^2 +$ OC$'^2$.

In an hyperbola, using the equation $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$, it will be found that $a^2 = x^2 - x'^2$, or (Fig. 2.) OH$^2 =$ OR$^2 -$ OR$'^2$ also that OM$^2 =$ OC$'^2 -$ OC^2.

PROPOSITION X.

78. In an ellipse the sum of the squares of two conjugate semi-diameters is equal to the sum, and in an hyperbola the difference between the squares of two conjugate semi-diameters is equal to the difference of the squares of the semi-axes of the curve.

Let EC, ZZ′ be the axes of the curve, AB and A′B′ any two conjugate diameters; it is required to prove that

$$OA^2 + OA'^2 = OE^2 + OZ^2.$$

The ordinates AR, &c. to the transverse axis CE being

Fig. 1.

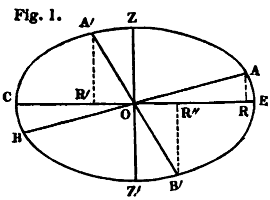
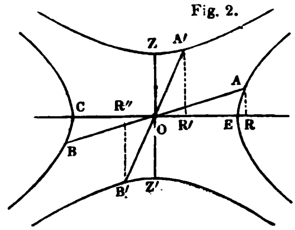


Fig. 2.



drawn; since, in the ellipse, (Fig. 1.) the semi-transverse and semi-conjugate axes being represented by t and c,

$$y^2 = \frac{c^2}{t^2}(t^2 - x^2) \qquad [(a) \text{ Art. 56.}]$$

we have $$OA^2 = \frac{c^2}{t^2}(t^2 - x^2) + x^2$$

and $$OA'^2 \text{ (or } OB'^2) = \frac{c^2}{t^2}(t^2 - x'^2) + x'^2,$$

substituting in the last equation $t^2 - x^2$ for x'^2 (Art. 77.),

$$OA'^2 = \frac{c^2}{t^2}\{t^2 - (t^2 - x^2)\} + t^2 - x^2.$$

Therefore

$$OA^2 + OA'^2 = t^2 + c^2. \quad (= OE^2 + OZ^2).$$

In like manner, using the equation

$$y^2 = \frac{c^2}{t^2}(x^2 - t^2), \qquad [(a) \text{ Art. 57.}],$$

it will be found that in an hyperbola (Fig. 2.)

$$OA^2 - OA'^2 = t^2 - c^2. \quad (= OE^2 - OZ^2).$$

PROPOSITION XI.

79. In an ellipse and an hyperbola the parallelogram formed on any two semi-conjugate diameters, as sides, is equal to the rectangle formed on the two semi-axes.

Fig. 1.

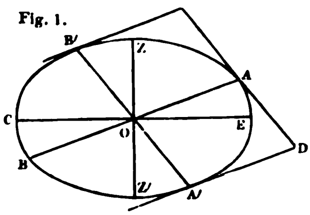
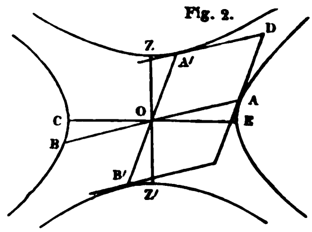


Fig. 2.



For the ellipse, let OE, OZ (Fig. 1.) be the semi-axes, OA and OA′, or OB, be two semi-conjugate diameters; also let the angle AOE $=\theta$ and EOA′ or EOB′ $=\theta'$.

Then [(*c*) Art. 76.]

$$\text{OA} = \frac{tc}{\sqrt{(t^2 \sin.^2\theta + c^2 \cos.^2\theta)}}, \text{ and OA}' \text{ or OB}' = \frac{tc}{\sqrt{(t^2 \sin.^2\theta' + c^2 \cos.^2\theta')}};$$

consequently

$$\text{OA} \cdot \text{OA}' \sin.(\theta+\theta') = \frac{t^2 c^2 \sin.(\theta+\theta')}{\sqrt{\{(t^2 \sin.^2\theta + c^2 \cos.^2\theta)(t^2 \sin.^2\theta' + c^2 \cos.^2\theta')\}}},$$

and (Pl. Trigon. Art. 75. *a.*) the first member of this equation is the equivalent of the parallelogram OD.

On multiplying together the terms under the radical sign, the first and last terms of the product, viz.

$$t^4 \sin.^2\theta \sin.^2\theta' \text{ and } c^4 \cos.^2\theta \cos.^2\theta'$$

are (Art. 74.) equal to one another; and therefore their sum may be considered as equal to twice the product of their square roots; that is to

$$2t^2c^2 \sin.\theta \sin\theta' \cos.\theta \cos.\theta'.$$

The sum of the other terms of the product under the radical sign is

$$t^2c^2 \sin.^2\theta \cos.^2\theta' + t^2c^2 \sin.^2\theta' \cos.^2\theta,$$

and the sum just mentioned is evidently equal to twice the product of the square roots of these terms. Thus the whole quantity under the radical sign is the square of the binomial

$$tc\,(\sin.\theta \cos.\theta' + \cos.\theta \sin.\theta'),$$

and the denominator of the fraction is (Pl. Trigon. Art. 31.) equivalent to

$$tc \sin.(\theta+\theta'). \qquad (a)$$

The value of OA.OA′ sin. $(\theta+\theta')$, that is of the parallelogram OD, is therefore tc, or the rectangle OE.OZ.

For the hyperbola, we have [(c'), Art. 76.], Fig. 2.,

$$OA=\frac{tc}{\sqrt{(c^2\cos.^2\theta-t^2\sin.^2\theta)}},\text{ and } OA'=\frac{tc}{\sqrt{(c^2\cos.^2\theta'-t^2\sin.^2\theta')}};$$

from which we obtain

$$OA\,.\,OA'\sin.(\theta'-\theta)=\frac{t^2c^2\sin.(\theta'-\theta)}{\sqrt{\{(c^2\cos.^2\theta-t^2\sin.^2\theta)(c^2\cos.^2\theta'-t^2\sin.^2\theta')\}}}.$$

Multiplying both numerator and denominator of the second member by $\sqrt{-1}$, that member becomes

$$\frac{t^2c^2\sin.(\theta'-\theta)\sqrt{-1}}{\sqrt{\{(t^2\sin.^2\theta-c^2\cos.^2\theta)(c^2\cos.^2\theta'-t^2\sin.^2\theta')\}}};$$

and simplifying the denominator as above, it becomes

$$tc\,(\sin.\theta'\cos.\theta-\cos.\theta'\sin.\theta),\text{ or } tc\sin.(\theta'-\theta).$$

Consequently, OA.OA′ sin. $(\theta'-\theta)=tc\sqrt{-1}$.

The imaginary term $\sqrt{-1}$ indicates merely that, in the hyperbola, the parallelogram OD is on the convex sides of the branches of the curve.

PROPOSITION XII.

80. To find, in an ellipse, the positions of two conjugate diameters which are equal to one another.

Let t and c represent the semi-axes of the ellipse, and a each of the conjugate semi-diameters; then (Art. 78.) we have

$$t^2+c^2=2a^2;$$

whence $$a^2=\tfrac{1}{2}(t^2+c^2).$$

But it is manifest, from the symmetry of the ellipse with respect to the transverse or conjugate axis, that two equal conjugate diameters must make equal angles with either of those axes; therefore, if θ represent the angle which each of the equal conjugate diameters makes with the transverse axis, we shall have [(a) Art. 79.]

$$a^2\sin.2\theta=tc;\text{ whence } a^2=\frac{tc}{\sin.2\theta}.$$

Equating, $$(t^2+c^2)\sin.2\theta=2tc,$$

and $$\sin.2\theta=\frac{2tc}{t^2+c^2},$$

whence θ may be found.

Or, since sin. $2\theta = 2$ sin. θ cos. θ (Pl. Trigon. Art. 34.)

$$= \frac{2 \sin.\theta \cos.^2\theta}{\cos.\theta}$$

$$= \frac{2 \tan.\theta}{\sec.^2\theta} \text{ or } = \frac{2 \tan.\theta}{1+\tan.^2\theta}.$$

Substituting this last value of sin. 2θ in the equation for sin. 2θ, we obtain, after reduction,

$$\tan.\theta = \frac{c}{t} \text{ or } \frac{t}{c}.$$

Thus the equal conjugate diameters of an ellipse are those which are parallel to lines joining the extremities of the transverse and conjugate axes of the curve.

It is evident from Art. 66. that when two conjugate diameters of an ellipse are equal to one another, the chord which is equal to the parameter of either diameter coincides with the conjugate diameter.

Since t tan. $\theta = c$ is the equation for an asymptote of an hyperbola, it is evident that, in the latter curve, the lines which correspond to the equal conjugate diameters in an ellipse are the asymptotes, or the two infinite diameters.

Proposition XIII.

81. If from one extremity of any diameter of an ellipse or hyperbola a chord be drawn parallel to any other diameter, then a straight line joining the other extremities of the chord and the first diameter will be parallel to a diameter which is conjugate to that other diameter.

Fig. 1.

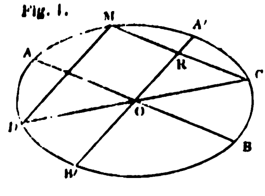
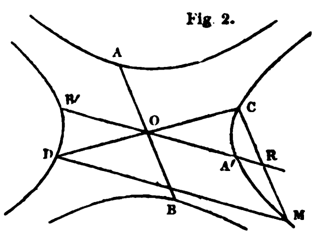


Fig. 2.



In an ellipse (Fig. 1.), let CD, AB be any two diameters, and let CM be a chord parallel to AB; then, if M, D be joined, MD will be parallel to A′B′, which is conjugate to AB.

Since CM is, by construction, parallel to AB, it is a double ordinate to A′B′, which is by hypothesis conjugate to AB;

therefore CM is bisected by A′B′ in R. But the diameter CD is bisected in O; consequently CM and CD are cut proportionally in R and O. Hence it follows (Euc. 2. VI.) that MD is parallel to A′B′.

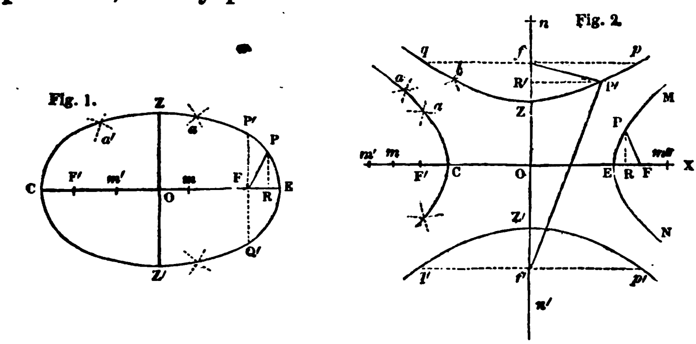
In a similar manner the proposition may be proved for an hyperbola, (Fig. 2.)

a. DEF. 14. Each of two chords drawn from the extremities of a diameter to the same point in the periphery of an ellipse or hyperbola is called a supplemental chord to the other; and chords thus drawn from the extremities of the transverse axis are called the principal supplemental chords.

PROPOSITION XIV.

82. To determine, in terms of rectangular co-ordinates, the length of a line drawn from one of the foci of an ellipse or hyperbola, or from the focus of a parabola, to any point in the curve.

Fig. 1. Fig. 2.



Let O (Fig. 1.) be the centre, EC the transverse, and ZZ′ the conjugate axis of an ellipse: let also F be a focus, and P any point in the curve, and imagine PR to be drawn perpendicular to EC.

Let $\text{OE}=t$, $\text{OZ}=c$, and OF (the excentricity) $=e$; then the co-ordinate axes, being supposed to coincide with the axes of the curve, we have $\text{OR}=x$, $\text{RP}=y$, and $\text{FR}=x-e$; or if FP is on the other side of the parameter, P′Q′, $\text{FR}=e\mp x$: the upper or the lower sign being used according as P is on the right or left of Z.

Now, (*a*) Art. 56., $$y^2=\frac{c^2}{t^2}(t^2-x^2);$$

or since $c^2=t^2-e^2$ (Art. 69.),

$$y^2=\frac{t^2-e^2}{t^2}(t^2-x^2)=\frac{t^4-t^2x^2-t^2e^2+e^2x^2}{t^2}.$$

Adding to the last fraction the value of FR², viz.

$$x^2 \mp 2ex + e^2, \text{ or } \frac{t^2x^2 \mp 2t^2ex + t^2e^2}{t^2},$$

we have

$$\text{FP}^2\ (=\text{FR}^2 + \text{RP}^2) = \frac{t^4 \mp 2t^2ex + e^2x^2}{t^2};$$

whence $$\text{FP} = \frac{t^2 \mp ex}{t} \text{ or } t \mp \frac{ex}{t}. \qquad (a)$$

When $t=1$, e' being put for the excentricity (=OF)

$$\text{FP} = 1 \mp e'x.$$

For an hyperbola (Fig. 2.), using the equation

$$y^2 = \frac{c^2}{t^2}\ (x^2 - t^2), \qquad [(a) \text{ Art. 57.}]$$

and, proceeding in like manner, OF or OF′ being represented by e, also $e^2 - t^2$, being equal to c^2 (Art. 71.) we have

$$\text{PF} = \frac{ex \mp t^2}{t}; \qquad (b)$$

and when $t=1$, $$\text{PF} = e'x \mp 1.$$

For a parabola, let it be supposed that MEN (Fig. 2.) is part of the curve, F being the focus; then, the co-ordinate axes being at right angles to one another, one of them coincident with EX, and the origin of the co-ordinates at E, we have

$$\text{ER} = x \text{ and } \text{RP} = y.$$

Putting p for the parameter of the axis we have (Art. 72.)

$$y^2\ (=\text{RP}^2) = px.$$

But (Art. 72.) $\text{EF} = \frac{p}{4}$; therefore $\text{FR} = \frac{p}{4} - x$; or $= x - \frac{p}{4}$,

and $\text{FR}^2 = \frac{p^2}{16} - \frac{px}{2} + x^2$:

adding, we have

$$\text{FP}^2\ (=\text{FR}^2 + \text{RP}^2) = \frac{p^2}{16} + \frac{px}{2} + x^2;$$

and $$\text{FP} = \frac{p}{4} + x. \qquad (c)$$

a. DEF. 15. A line drawn from a focus to any point of an ellipse, an hyperbola, or a parabola, is sometimes called a radius vector.

83. COR. In an ellipse, the sum of two lines drawn from the foci to any point in the curve; and, in an hyperbola, the

difference of two lines so drawn, is equal to the transverse axis.

For, in the ellipse [(a) Art. 82.] $\text{FP}=\dfrac{t^2-ex}{t}$,

and $\text{F}'\text{P}=\dfrac{t^2+ex}{t}$;

therefore $\text{F}'\text{P}+\text{FP}=2t$.

In the hyperbola [(b) Art. 82.] $\text{FP}=\dfrac{ex-t^2}{t}$,

and $\text{F}'\text{P}=\dfrac{ex+t^2}{t}$;

therefore $\text{F}'\text{P}-\text{FP}=2t$.

a. If the conjugate axis ZZ' (Fig. 2.) of an hyperbola be produced in opposite directions, and if double ordinates, pq, $p'q'$, each equal to the parameter of that axis be taken, it will be found that f and f', the points in which those ordinates cut the axis produced, are each at a distance from O, the centre, equal to OF or OF′.

For (Art. 65.), fp, or $f'p'$, being half the parameter of ZZ',

$$\text{OZ} : \text{OE} :: \text{OE} : fp \text{ or } f'p';$$

whence fp, or $f'p'$, $=\dfrac{t^2}{c}$ and $fp^2=\dfrac{t^4}{c^2}$. (a)

But [(b′) Art. 57.], $fp^2=\dfrac{t^2}{c^2}(\text{O}f^2-c^2)$;

or, representing $\text{O}f$ by e,

$$fp^2=\frac{t^2}{c^2}(e^2-c^2). \qquad (b)$$

Equating the second members of (a) and (b), we have

$$e^2-c^2=t^2, \text{ or } e^2=c^2+t^2;$$

therefore $\text{O}f$ or $\text{O}f'$ is (Art. 71.) equal to the excentricity OF or OF′ on the transverse axis. The points f and f' are called conjugate foci.

b. If lines be drawn from f and f' to any point P′ in either of the conjugate hyperbolæ, it will be found that the difference between those lines is equal to the conjugate axis;

or $$f'\text{P}'-f\text{P}'=\text{ZZ}'.$$

For OR' being represented by y and $\text{O}f$ or $\text{O}f'$ by e,

$$\text{R}'\text{P}'^2 \text{ [(b′) Art. 57.]} =\frac{t^2}{c^2}(y^2-c^2);$$

and $\text{R}'f=e-y$, or $\text{R}'f^2=(e-y)^2$:

therefore $f\text{P}'^2 = \frac{t^2}{c^2}(y^2 - c^2) + (e - y)^2$;

or, substituting $e^2 - c^2$ for t^2 (Art. 71.), and reducing

$$f\text{P}'^2 = \frac{e^2y^2 - 2c^2ey + c^4}{c^2};$$

whence $f\text{P}' = \frac{ey - c^2}{c}$.

In like manner $f'\text{P}' = \frac{ey + c^2}{c}$;

consequently $f'\text{P}' - f\text{P}' = 2c$ $(= \text{ZZ}')$.

Scholium. On this proposition depends one of the methods of determining any number of points in the periphery of an ellipse, or in the curve of an hyperbola. Let CE (Fig. 1.) be the transverse axis of an ellipse; F and F′, the two foci; and between F and F′ in the same axis take any point *m*: then, with F as a centre, and a radius equal to E*m*, describe a circular arc; also, with F′ as a centre, and a radius equal to C*m*, describe a circular arc: this will intersect the former arc in some point, as *a*. In like manner, taking other points as *m′*, and describing arcs from F and F′ as centres, with radii equal to E*m′* and C*m′* respectively, there will be found other intersections, as *a′*. The points C, *a*, *a′*, &c. will, by the corollary (Art. 83.), be in the periphery of an ellipse; and a curve line may be drawn through the points so found.

Again, let CE (Fig. 2.) be the transverse axis of an hyperbola, F and F′ the foci; and beyond F, or F′, take any point, as *m*, on the axis CE produced. Then, with F as a centre, and a radius equal to E*m*, describe an arc; also, with F′ as a centre, and a radius equal to C*m*, describe an arc crossing the former in *a*. In like manner, taking other points, as *m′*, and describing arcs from F and F′ as centres, with radii equal to E*m′* and C*m′*, there may be found other intersections, as *a′*; the points C, *a*, *a′*, &c. will, by the corollary (Art. 83.), be in the curve line of an hyperbola. The corresponding branch, MEN, may be formed by taking points, as *m″*, beyond F in the line CE produced.

If, on ZZ′ produced there be taken points, as *f*, *f′*, at distances from O equal to OF or OF′, the branches *q*Z*p*, *q′*Z′*p′* of the conjugate hyperbola may be described by means of points, as *n*, *n′*, taken beyond *f* or *f′*. Thus, from *f* as a centre, with a radius equal to Z*n*, describe an arc, and from *f′* as a centre, with a radius equal to Z′*n*, describe an arc intersecting the former; the point *b* of intersection will [Art. 83. *b*.] be in the branch *q*Z*p*.

An ellipse may be described by setting up, in a board, pins at two points, as F and F′, at a distance from one another equal to the interval between the two foci of the required ellipse, and attaching to each of them one of the ends of a thread whose length is equal to CE, the transverse axis: then, a pencil being kept in a bend of the thread, and moved while the two parts of the string are tight, it will describe the periphery of the ellipse.

PROPOSITION XV.

84. To determine, in terms of polar co-ordinates, the length of a line drawn from a focus to any point in an ellipse, an hyperbola, and a parabola.

Fig. 1.

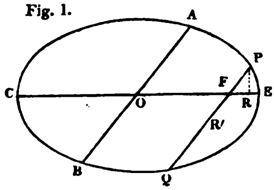
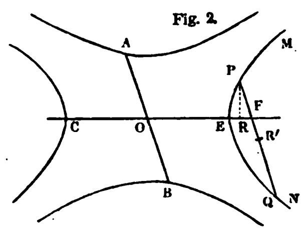


Fig. 2.



Let CE (Fig. 1.) be the transverse axis of an ellipse, O the centre, F a focus, or pole, and P any point in the curve; also, let the angle EFP be represented by θ, and OF by e; then, in the figure,

$$x\ (=\text{OR})\ \text{becomes}\ e+\text{PF}\cos.\theta.$$

Substituting for PF its value $\dfrac{t^2-ex}{t}$ (Art. 82.)

we have $$x=e+\frac{t^2-ex}{t}\cos.\theta;$$

whence $$tx=te+(t^2-ex)\cos.\theta,$$

and $$x=\frac{te+t^2\cos.\theta}{t+e\cos.\theta}.$$

Substituting this value of x in that of PF, we have

$$\text{PF}\left(=t-\frac{ex}{t}\right)=\frac{t^2-e^2}{t+e\cos.\theta}=\frac{c^2}{t+e\cos.\theta}.$$

When for θ in this expression, $180°+\theta$ is substituted; since $\cos.(180°+\theta)=-\cos.\theta$, we get

$$\text{FQ}=\frac{c^2}{t-e\cos.\theta}.$$

Hence $\qquad$ $PF+FQ=\dfrac{2tc^2}{t^2-e^2\cos.^2\theta}$.

For an hyperbola, we have (Fig. 2.)

$$x\,(=OR)=e-PF\cos.\theta;$$

therefore, substituting the value of PF, viz. $\dfrac{ex-t^2}{t}$ (Art. 82.), and proceeding as above, we get

$$PF=\frac{e^2-t^2}{t+e\cos.\theta},\text{ or }\frac{c^2}{t+e\cos.\theta},$$

the same expression as for the radius vector of an ellipse.

For a parabola, considering MEN (Fig. 2.) as part of the curve, we have (Art. 82.), p being the parameter of the axis,

$$FR=\frac{p}{4}-x;$$

and the angle EFP being represented by θ,

$$FR\,(=FP\cos.\theta)=(\frac{p}{4}+x)\cos.\theta.$$

Equating these values of FR, we get

$$\frac{p}{4}-\frac{p}{4}\cos.\theta=x+x\cos.\theta;$$

whence $\qquad$ $\dfrac{p}{4}\cdot\dfrac{1-\cos.\theta}{1+\cos.\theta}=x.$

Substituting this value of x in $\dfrac{p}{4}+x$, the equivalent of PF, we have

$$PF=\frac{p}{4}+\frac{p}{4}\cdot\frac{1-\cos.\theta}{1+\cos.\theta};$$

which, reduced, gives

$$PF=\frac{p}{2\,(1+\cos.\theta)}.$$

For θ substituting $180^\circ+\theta$, we get

$$FQ=\frac{p}{2\,(1-\cos.\theta)}:$$

Hence $PF+FQ=\dfrac{p}{1-\cos.^2\theta}=\dfrac{p}{\sin.^2\theta}$.

85. COR. 1. If any chord be drawn through a focus of an ellipse or hyperbola, half that chord is a third proportional to the semi-transverse axis, and the semi-diameter which is parallel to the chord.

Let PQ (Figs. 1. and 2.) be a chord drawn through the focus, and let AB be a diameter of the curve, drawn parallel to the chord: then, as above, both in the ellipse and hyperbola,

$$\text{PQ} = \frac{2tc^2}{t^2 - e^2 \cos.^2 \theta};$$

therefore, PQ being bisected in R′,

$$\text{PR}' = \frac{tc^2}{t^2 - e^2 \cos.^2 \theta}.$$

But the angle AOE being equal to PFE, or its supplement, we have [(a′) Art. 76.]

$$\text{AO}^2 = \frac{t^2 c^2}{t^2 - e^2 \cos.^2 \theta}:$$

therefore $\frac{\text{AO}^2}{t} \left(= \frac{\text{AO}^2}{\text{OE}}\right) = \text{PR}'$,

or OE : AO :: AO : PR′.

86. COR. 2. Taking the reciprocals of the values of PF and FQ, and adding them together, we have, in the ellipse and hyperbola,

$$\frac{1}{\text{PF}} + \frac{1}{\text{FQ}} = \frac{2t}{c^2};$$

and, in a parabola, $\frac{1}{\text{PF}} + \frac{1}{\text{FQ}} = \frac{4}{p}.$

Thus, in the three curves, the sum of the reciprocals of the two segments made at the focus on any chord passing through that point is constant; and is equal to the reciprocal of one quarter of the parameter of the axis on which the focus is situated.

PROPOSITION XVI.

87. In an ellipse, or an hyperbola, the rectangle contained by two lines drawn from the foci, to meet at any point in the curve, is equivalent to the square of the semi-diameter, which is conjugate to one passing through the point.

Fig. 1.

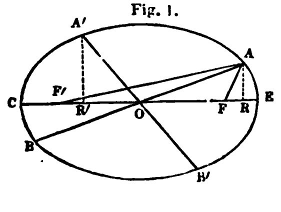
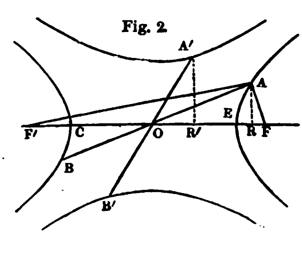


Fig. 2.



In the ellipse and hyperbola let F and F′ be the foci; let AF and AF′ be two lines drawn to any point A in the curve; also let A′B′ be a diameter conjugate to AB: then, O being the centre,

$$\text{AF}.\ \text{AF}' = \text{A}'\text{O}^2.$$

The axes of the co-ordinates being coincident with the transverse and conjugate axes of the curves, imagine AR, A′R′ to be drawn perpendicular to the transverse axis EC; let OR $=x$, OR′ $=x'$; then t and c representing the semi-axes of the curve, we have, for the ellipse [(*a*) Art. 82.] Fig. 1,

$$\text{AF} = \frac{t^2 - ex}{t},\quad \text{AF}' = \frac{t^2 + ex}{t};$$

therefore $\quad \text{AF}.\ \text{AF}' = \dfrac{t^4 - e^2 x^2}{t^2}.$

Now [(*a*) Art. 75.], $\text{A}'\text{O}^2 = \dfrac{t^2 c^2 + (t^2 - c^2) x'^2}{t^2}$;

and putting (Art. 77.) $t^2 - x^2$, for x'^2, we have, after reduction $\text{A}'\text{O}^2 = \dfrac{t^4 - (t^2 - c^2) x^2}{t^2}$, or $= \dfrac{t^4 - e^2 x^2}{t^2}$,

which is identical with the value of AF. AF′.

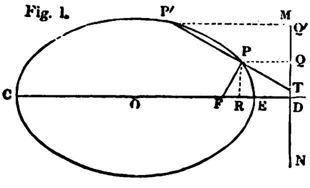
In like manner may the proposition be proved for the hyperbola, Fig. 2.

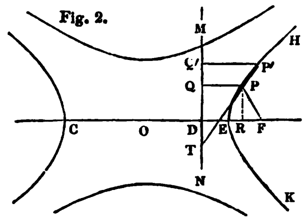
If AO, and A′O, coincide with the semi-axes of the curve, the proposition is manifest, for then (Art. 83.) AF and AF′, being equal to one another, will be each equal to the semi-transverse axis.

Proposition XVII.

88. If a straight line be drawn perpendicular to the transverse axis of an ellipse or hyperbola, intersecting that axis in a point at a distance from the centre of the curve, which is a third proportional to the excentricity and the semi-transverse axis; the length of a line drawn from a focus to any point in the curve is, to the perpendicular distance from that point to the line first drawn, as the excentricity is to the semi-transverse axis. Also, if a straight line be drawn perpendicular to the axis of a parabola, at a distance from its extremity equal to the distance of

the focus from the same point; the length of a line from the focus to any point in the curve is equal to the perpendicular distance from that point to the line first drawn.





Let EC (Figs. 1. and 2.) be the transverse axis of an ellipse or hyperbola, O its centre, F a focus, and P any point in the curve; also in CE, produced if necessary, let a point D be taken so that

$$\text{OF} : \text{OE} :: \text{OE} : \text{OD},$$

and through D a line MN be drawn perpendicular to CE; then joining F, P, and drawing PQ perpendicular to MN, it will be found that

$$\text{OF} : \text{OE} :: \text{FP} : \text{PQ}.$$

Let t represent the semi-transverse axis, e the excentricity, and x the abscissa OR corresponding to P; then (Art. 82.)

in the ellipse, $\text{FP} = \dfrac{t^2 \mp ex}{t}$:

but by hypothesis, $e : t :: t : \dfrac{t^2}{e}$ $(=\text{OD})$,

and consequently RD, or PQ,

$$= \frac{t^2}{e} \mp x, \text{ or } \frac{t^2 \mp ex}{e}:$$

the negative sign being used when PR is between O and E, and the positive sign when it is between O and C.

Therefore $\quad \text{FP} : \text{PQ} :: \dfrac{1}{t} : \dfrac{1}{e}$,

or $\quad \text{FP} : \text{PQ} :: e : t.$

The like result will be obtained for the hyperbola, on putting $\dfrac{ex \mp t^2}{t}$ for PF (Fig. 2.), the equivalent of RD being $\dfrac{ex \mp t^2}{e}$. In this curve the line MN falls between O and E;

and, in both curves there is also a line corresponding to MX on the opposite side of O.

a. For a parabola, supposing HEK (Fig. 2.) to be part of such a curve, ER now representing x, and p being the parameter of the axis,

$$\text{PF } ((c)\ 82.) = \frac{p}{4} + x:$$

but by hypothesis, DE = EF and (Art. 72.) $\text{EF} = \frac{p}{4}$;

therefore $\text{DR, or QP,} = \frac{p}{4} + x;$

consequently FP = PQ.

89. DEF. 16. A line drawn perpendicular to the transverse axis of an ellipse or hyperbola, or to the axis of a parabola, and at such a distance from the vertex of either curve, that lines drawn from a focus to any points in the curve, and others from those points perpendicularly to the first line have a constant ratio to one another, is called a directrix. Such is the line MN in the above figures.

a. COR. Let P and P′ be any two points in the curve of an ellipse, hyperbola, or parabola, and let them be joined by a line, as P′PT, which, being produced, will cut the directrix in some point as T; also let P′Q′ be drawn perpendicularly to the directrix. Then, since by the proposition,

FP : PQ :: FP′ : P′Q′,

(:: e : t in the two first curves, and a ratio of equality in the parabola)

or FP : FP′ :: PQ : P′Q′,

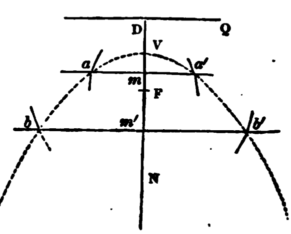
and by the similarity of the triangles PTQ, P′TQ,

PQ : P′Q′ :: PT : P′T;

it follows that, in all the three curves,

FP : FP′ :: PT : P′T.

Scholium. On Art. 88. (*a*), depends the method of finding, geometrically, any number of points in the curve of a parabola. Let VN be the direction of the axis of a parabola, V the vertex, and F the focus; also, VD being made equal to VF, let DQ, drawn perpendicularly to VN, be the directrix. Take any number of points, as m, m' on VN, and through them draw lines perpendicularly to VN: then, with F as a centre and radii equal to Dm



and Dm', respectively, describe arcs intersecting those lines in a and a', b and b'. The points V, a, b, a', &c. will be in a parabola, and through them the curve may be drawn.

If a focus, the position of the directrix, and one extremity of the transverse axis of an ellipse, hyperbola or parabola be given, the curve may be described. For, let F (Figs. to Art. 88.) be the focus, E one extremity of the axis, and MN the directrix cutting the transverse axis, produced or not, in D; then (Art. 88.) for the ellipse and hyperbola,

$$t : e :: \text{ED} : \text{EF} :: \text{CD} : \text{CF};$$

therefore

$$\text{ED}-\text{EF} \text{ (or EF}-\text{ED)} : \text{EF} :: \text{CD}-\text{CF} \text{ (or CF}-\text{CD)} : \text{CF}.$$

Thus CF, and consequently CE, the transverse axis, is found; and the ellipse or hyperbola may be described by the Scholium Art. 83.

The parabola may be described by the Scholium to this Article.

PROPOSITION XVIII.

90. To find the equation for a tangent to an ellipse, hyperbola, or parabola.

Fig. 1

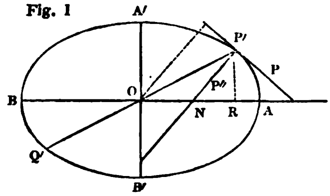
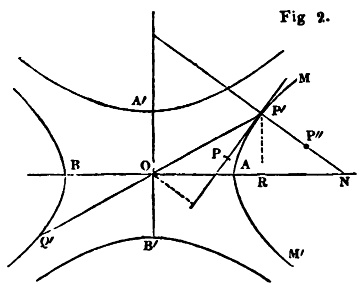


Fig 2.



Let PP$'$ be a tangent to any one of the curves, P$'$ being the point of contact: let OA, OA$'$, any two semi-conjugate diameters of an ellipse or hyperbola, be the co-ordinate axes, and let P$'$Q$'$ be a diameter passing through the point of contact. Let a and b represent those semi-diameters, x', y' the co-ordinates of P$'$ and x, y the co-ordinates of any other point P in the tangent.

Then, since (Art. 38.) PP$'$ is parallel to a diameter which is conjugate to OP$'$, if the equation of the latter be represented by $qx'=y'$; the equation for PP$'$ will be (Art. 15. and (a) Art. 73.), for the ellipse (fig. 1.),

$$-\frac{b^2}{a^2q}(x'-x)=y'-y, \text{ or } \frac{b^2}{a^2q}(x'-x)=y-y'. \quad (a)$$

Substituting in this last the value of q from the preceding equation, we have

$$\frac{b^2 x'}{a^2 y'}(x'-x)=y-y':$$

hence $$b^2 x'^2 - b^2 x' x = a^2 y' y - a^2 y'^2,$$

or $$a^2 y y' + b^2 x x' = a^2 y'^2 + b^2 x'^2;$$

but the second member of this equation is [(b) Art. 56.] equivalent to $a^2 b^2$. Therefore the required equation for the tangent to an ellipse, when referred to any two conjugate diameters, is

$$a^2 y y' + b^2 x x' = a^2 b^2. \qquad (a')$$

In an hyperbola (fig. 2.) the equation for PP′ will be (Art. 15. and (b) Art. 73.)

$$\frac{b^2}{a^2 q}(x'-x) = y'-y; \qquad (b)$$

and, substituting in it the above value of q, there is obtained the equation,

$$b^2 x'^2 - b^2 x' x = a^2 y'^2 - a^2 y' y,$$

or [(b) Art. 57.] $a^2 y' y - b^2 x' x = -a^2 b^2. \qquad (b')$

In a parabola, MAM (fig. 2.) being considered as a portion of such curve, since PP′ is parallel to a chord which is bisected by a diameter passing through P′, the equation for that tangent will be (Art. 15. and (c′) Art. 73.),

$$\frac{b^2}{2 a y'}(x'-x) = y'-y; \qquad (c)$$

whence $$b^2 x' - b^2 x = 2 a y'^2 - 2 a y y':$$

or, substituting for y'^2 its equivalent $\frac{b^2}{a} x'$ [(a) Art. 63.],

$$b^2 x' + b^2 x = 2 a y y'; \qquad (c')$$

or again, since $\frac{b^2}{a} = p$ (p representing the parameter of the diameter passing through A)

$$p x' + p x = 2 y y'. \qquad (c'')$$

91. DEF. 17. A straight line as P′N drawn from any point in a curve perpendicularly to a tangent at that point is called a normal.

92. COR. to the Proposition. To find the equation for a normal to an ellipse, an hyperbola, or a parabola; the co-ordinate axes being rectangular.

A normal being at right angles to a tangent to the curve at the point of contact; from the equation (a) for the tangent, we have [(a) Art. 17. and (a) Art. 73.], the co-ordinate axes coinciding with the transverse and conjugate axes, for the equation of the normal, in an ellipse,

$$\frac{t^2 q}{c^2}(x'-x)=y'-y, \qquad (d)$$

in this equation x' and y' are co-ordinates of the point of contact, and x, y are co-ordinates of any point as P'' in the normal.

Substituting $\frac{y'}{x'}$ for q, this equation becomes

$$t^2 y'x' - t^2 y'x = c^2 x'y' - c^2 x'y. \qquad (d')$$

From the equation (b) we have [(a) Art. 17.] for the equation of a normal in an hyperbola, the co-ordinate axes coinciding with the axes of the curve,

$$-\frac{t^2 q}{c^2}(x'-x)=y'-y; \qquad (e)$$

and substituting the value of q as before, the equation becomes

$$t^2 y'x - t^2 y'x' = c^2 x'y' - c^2 x'y. \qquad (e')$$

From the equation (c) we have [(a) Art. 17.] for the equation of a normal in a parabola, the co-ordinate axes being rectangular.

$$-\frac{2ty'}{c^2}(x'-x)=y'-y,$$

or $$2ty'x - 2ty'x' = c^2 y' - c^2 y; \qquad (f)$$

or again $$2y'x - 2y'x' = p(y'-y): \qquad (f')$$

p being the parameter of the axis.

PROPOSITION XIX.

93. If a tangent be drawn at any point in the curve line of an ellipse or hyperbola, and be produced both ways till it cut two conjugate diameters, also if ordinates be drawn from the point of contact to both diameters, the semi-diameter on which each ordinate falls will be a mean proportional between the distances, on that semi-diameter produced, of the ordinate and tangent from the centre of the curve.

Let TT′ be a tangent to the curve, at P′, meeting the conjugate diameters AB and A′B′, either of them being pro-

Fig. 1.

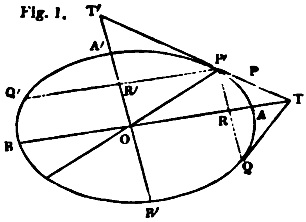
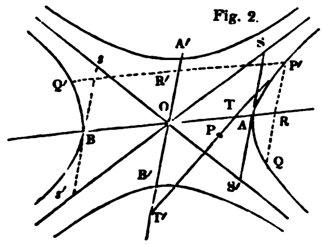


Fig. 2.



duced, if necessary, in T and T′; and, the co-ordinate axes coinciding with these diameters, let P′R, P′R′ be the ordinates; then, O being the centre of the curve, in both figures,

$$\text{OR} : \text{OA} :: \text{OA} : \text{OT},$$

and

$$\text{OR}' : \text{OA}' :: \text{OA}' : \text{OT}'.$$

For both curves let the conjugate semi-diameters be represented by a and b: let OR, RP′ be represented by x' and y', and let x, y be co-ordinates of any other point P in the tangent. Then [(a') Art. 90.] the equation for the tangent to an ellipse (Fig. 1.) is

$$a^2yy' + b^2xx' = a^2b^2;$$

but at the point T, where $y = 0$, the equation becomes

$$b^2xx' = a^2b^2, \text{ or } xx' = a^2.$$

Hence $\quad x' : a :: a : x$, or $\text{OR} : \text{OA} :: \text{OA} : \text{OT}$;

and $$\text{OT} = \frac{a^2}{x'}.$$

At the point T′ we have $x = 0$, and the equation then becomes

$$a^2yy' = a^2b^2, \text{ or } yy' = b^2:$$

whence $\quad y' : b :: b : y$, or $\text{OR}' : \text{OA}' :: \text{OA}' : \text{OT}'$;

and $$\text{OT}' = \frac{b^2}{y'}.$$

For an hyperbola (Fig. 2.), the equation being [(b') Art. 90.]

$$b^2xx' - a^2yy' = a^2b^2:$$

on making first $y = 0$ and afterwards $x = 0$, we have respectively,

$$xx' = a^2 \text{ and } -yy' = b^2;$$

whence $\quad x' : a :: a : x$, or $x = \dfrac{a^2}{x'}$ $(= \text{OT})$,

and $\quad y' : b :: b : -y$, or $y = -\dfrac{b^2}{y'}$ $(\ \ \text{OT}')$.

94. COR. 1. It follows from this proposition that, in the ellipse,

$$\frac{a^2}{x'}-x', \text{ or } \frac{a^2-x'^2}{x'}, = \text{RT}, \text{ and } \frac{b^2}{y'}-y', \text{ or } \frac{b^2-y'^2}{y'}, = \text{R}'\text{T}';$$

and, in the hyperbola,

$$x'-\frac{a^2}{x'}, \text{ or } \frac{x'^2-a^2}{x'}, = \text{RT}, \text{ and } y'+\frac{b^2}{y'}, \text{ or } \frac{y'^2+b^2}{y'}, = \text{R}'\text{T}'.$$

95. COR. 2. From the above values of OT and OT′ for the hyperbola (since x' in the one and y' in the other may be infinite) we have, when the point of contact is infinitely remote from the origin of the co-ordinates, OT$=0$, and OT′$=0$. That is, in an hyperbola, when the point of contact is so situated, the tangent, which then [Art. 52. *b*.] becomes an asymptote, will pass through the centre of the curve: such a tangent is, therefore, one of the infinite diameters of the curve.

96. COR. 3. The equation for an asymptote, when referred to any pair of conjugate diameters, is (Art. 59.)

$$\pm\frac{b}{a}x=y;$$

and, when $x=a$, $y=\pm b$. Now y is an ordinate to one of the diameters (as AB, fig. 2., above), therefore it is parallel to P′R, or it is then a tangent to the curve, at A or B: thus, and on account of the double sign prefixed to b, if a tangent be drawn at either extremity of a diameter of an hyperbola, the four segments, AS, AS′, B*s*, B*s*′, intercepted between the points of contact A and B, and both the asymptotes, are equal to one another: each segment being equal and parallel to the diameter, which is conjugate to AB. See Art. 60.

97. COR. 4. In an ellipse and an hyperbola, if P′R and P′R′ be produced to Q and Q′, it is manifest, since both OR and OR′ remain the same, and P′R$=$RQ, also P′R′$=$R′Q′, that a tangent drawn from Q, and one from Q′, would meet the diameters BA and B′A′ respectively, either of these being produced, if necessary, in T and T′.

98. COR. 5. If any number of ellipses or hyperbolas be formed on the same axis, as EC, and if points, P, P′, &c. in the

Fig 1.

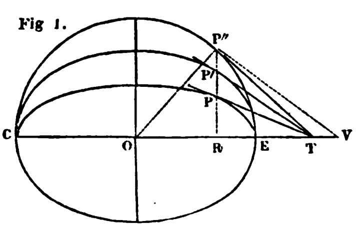
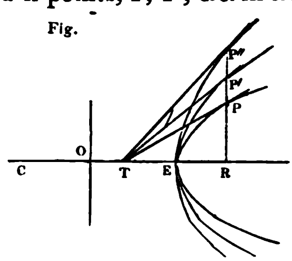


Fig.



curves be taken in the direction of the same ordinate RP to that. axis, the straight lines touching the curves at those points will all meet in one point T on that axis.

For in the ellipses and hyperbolas (Art. 93.),

$$\text{OR} : \text{OE} :: \text{OE} : \text{OT};$$

and the three first terms being constant, the last is constant.

If one of the curves, as CP″E (Fig. 1.), be a semi-circle, the line touching it at P″ will also meet CE, produced, in T.

For if the tangent at P″ do not meet CE in T, let it meet it in some other point, as V: then, the triangle OP″V being right angled at P″, and P″R being perpendicular to OV, we have (EUC. Cor. 8. VI.)

$$\text{OR} : \text{OP}''(=\text{OE}) :: \text{OP}''(=\text{OE}) : \text{OV};$$

but, by the proposition, OR : OE :: OE : OT.
Therefore OV = OT, or the tangent to the circle meets OE produced in T.

99. COR. 6. In ellipses and hyperbolas formed on the same axis, as CE, the circle CP″E being included among the former, the several ordinates PR, P′R, &c. are to one another in the same proportions as the semi-axes which are parallel to them.

For the equations of the ellipses are [(a) Art. 56.]

$$y = \frac{c}{t}(t^2 - x^2)^{\frac{1}{2}},\ y' = \frac{c'}{t}(t^2 - x^2)^{\frac{1}{2}},\ \&c.$$

and that of the circle CP″E may be represented by

$$y' = \frac{t}{t}(t^2 - x^2)^{\frac{1}{2}};$$

in which equations all the quantities, except those represented by y, and by the numerators of the fractions, which are coefficients of $(t^2 - x^2)^{\frac{1}{2}}$, are constant: thus the several ordinates vary with the semi-conjugate axes.

100. COR. 7. If a tangent be drawn at any point P′ (figures to the Proposition) in an ellipse or hyperbola to meet any two conjugate diameters AB, A′B′ of the curve, and if ordinates, P′R, P′R′, be drawn to those diameters from the same point,

$$\text{AR} \,.\, \text{RB} = \text{OR} \,.\, \text{RT}.$$

For OR put a, for OA′ put b, for OR or R′P′ put x, and for RP′ or OR′ put y; then we shall have

$$\text{AR} = a - x, \text{ and } \text{BR} = a + x;$$

therefore AR . RB $= a^2 - x^2$.

Again, OR $=x$, and (Art. 94.) RT $=\frac{a^2}{x}-x$, or $\frac{a^2-x^2}{x}$;

therefore OR . RT $=a^2-x^2$.

Thus AR . RB $=$ OR . RT.

In like manner it might be proved that

A′R′ . R′B′ $=$ OR′ . R′T′.

101. COR. 8. If a tangent be drawn as before,

OR . OT $=$ AO^2, and OR′ . OT′ $=$ OA′^{2}:

For

OR $=x$, and (Art. 93.) OT $=\frac{a^2}{x}$; therefore OR . OT $=a^2$ ($=$ AO^2).

Also OR′ $=y$, and OT′ $=\frac{b^2}{y}$; therefore OR′ . OT′ $=b^2$ ($=$ A′O^2).

102. COR. 9. If a tangent be drawn as before,

BR . AT $=$ AR . BT.

For BR $=a+x$, and AT $=\frac{a^2}{x}-a$, or $\frac{a^2-ax}{x}$, or $\frac{a(a-x)}{x}$;

therefore BR . AT $=\frac{a(a^2-x^2)}{x}$.

Again, AR $=a-x$, and BT $=\frac{a^2}{x}+a$, or $\frac{a(a+x)}{x}$;

therefore AR . BT $=\frac{a(a^2-x^2)}{x}$.

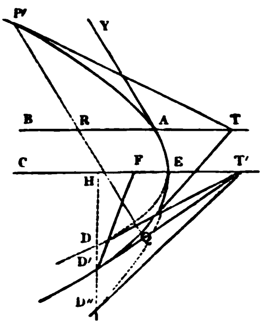
Thus BR . AT $=$ AR . BT.

103. DEF. 18. The space on any diameter of a curve between the points at which that diameter, produced if necessary, is intersected by a tangent, and an ordinate drawn from the point of contact, is called a subtangent.

PROPOSITION XX.

104. If a tangent be drawn at any point in the curve of a parabola so as to cut any diameter produced, also if an ordinate to that diameter be drawn from the point of contact, the distance on the diameter, from the ordinate to the extremity of the diameter, is equal to the distance from that extremity to the tangent.

Let P′T be a tangent to the curve, at P′, meeting the diameter BA produced in T; also, the co-ordinate axes coinciding with that diameter, and with a tangent AY at A, let P′R be an ordinate drawn from P′; then



$$RA = AT.$$

Let x', y', be the co-ordinates of P′, and x, y those of any other point in the tangent; also let p represent the parameter of the diameter AB: then [(c'') Art. 90.] the equation for the tangent is

$$p\,(x'+x)=2yy':$$

and at the point T, where $y=0$,

$$x'+x=0, \text{ or } x=-x':$$

that is, $AT=AR$, or $RT=2AR$.

It is manifest that, if P′R were produced to Q, since AR remains the same, and $RP'=RQ$, a tangent drawn from Q would meet BA produced, in T.

105. COR. 1. If any number of parabolas be formed on a common axis, EC, and if points D, D′, D″, &c. in the curves be taken, in the direction of the same ordinate HD to that axis; the straight lines touching the curves at those points will all meet in one point T′ on the same axis.

For, by the proposition, in all the parabolas,

$$HE=ET';$$

and HE is constant; therefore ET′ is constant.

106. COR. 2. In all the parabolas, on the same axis, the several ordinates corresponding to any one abscissa are to one another as the square roots of the parameters.

For y, y', &c. being the ordinates, x the common abscissa, and p, p', &c. the parameters of the axes for the different curves; we have (Art. 72.)

$$y=p^{\frac{1}{2}}\,x^{\frac{1}{2}},\ y'=p'^{\frac{1}{2}}\,x^{\frac{1}{2}}, \text{ \&c.}$$

in all which equations, x being constant, y varies with $p^{\frac{1}{2}}$; therefore, &c.

107. COR. 3. If a tangent be drawn from any point, as D′ in a parabola, to meet the axis of the curve produced, as in T′; the distance of the focus from the point of contact is equal to the distance, on the axis, from the focus to the point in which the tangent meets the axis.

Let F be the focus; then $FD'=FT'$.

For p being the parameter of the axis, and EH $(=\text{ET}')$ being represented by x,

$$\text{FD}'[(c)\text{ Art. } 82.]=\frac{p+4x}{4};$$

also $\quad \text{FE}=\frac{p}{4}$; therefore $\text{FT}=\frac{p}{4}+x$, or $\frac{p+4x}{4}$.

Thus $\quad \text{FD}'=\text{FT}'$.

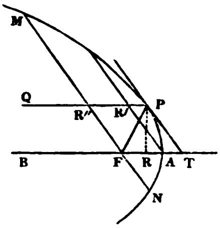
It follows that the angle $\text{FT}'\text{D}'=\text{FD}'\text{T}'$.

PROPOSITION XXI.

108. A line drawn from the focus of a parabola, to any point in the curve, is equal to one quarter of the parameter of the diameter passing through that point.

Let AB be the axis, F the focus, P any point in the curve, and PQ the diameter passing through P; then, if p' be the parameter of PQ,

$$\text{FP}=\tfrac{1}{4}p'.$$



Draw the ordinate PR to the axis, and let PT be a tangent at P meeting BA produced in T; also draw AR′, an ordinate to PQ, from A the vertex. Let p be the parameter of the axis, and let AR be represented by x: then (Art. 104.) $\text{RT}=2x$, and (Art. 72.) $\text{RP}^2=px$;

therefore, PT^2, or $\text{R}'\text{A}^2$, $=px+4x^2$, or $=x\,(p+4x)$.

But (Euc. 34. I.) $\text{R}'\text{P}=\text{AT}=$(Art. 104.) AR, or x;

therefore (Art. 72.), $\quad \text{R}'\text{A}^2=p'x$.

Consequently, $p'x=(p+4x)\,x$, or $p'=p+4x$.

But $[(c)$ Art. 82.$]$ $\text{FP}=\frac{p+4x}{4}$; therefore $\text{FP}=\frac{1}{4}p'$.

109. COR. If a double ordinate, as MN, of any diameter, as PQ intersecting PQ in R″, be equal to the parameter of that diameter, it will pass through F, the focus.

For (Art. 72.) $\quad \text{PR}''=\frac{1}{4}p'$;

and, by this proposition, $\frac{1}{4}p'=\text{FP}$

also (Art. 107.) $\quad \text{FP}=\text{FT}$;

therefore $\quad \text{PR}''=\text{FT}$.

But MN, being a double ordinate of the diameter PQ, is parallel to PT; therefore MN passes through F.

PROPOSITION XXII.

110. If a normal be drawn at any point in the curve of an ellipse, or hyperbola, to cut either of the axes, and if an ordinate be drawn from the point to that axis, the square of the semi-axis on which the normal falls will be to the square of the conjugate semi-axes as the distance of the ordinate from the centre of the curve, on the first axis, is to the distance between the ordinate and the normal.

Fig. 1.

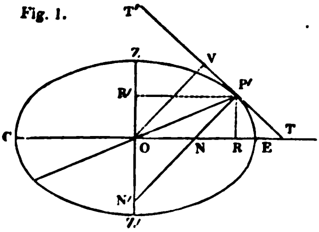
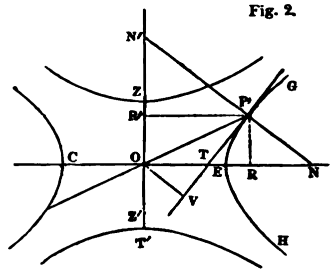


Fig. 2.



Let P′NN′, NP′N′, be, respectively, an ordinate to an ellipse and an hyperbola at the point P′, meeting the axes EC, ZZ′ in N and N′, and let the ordinates P′R, P′R′ be drawn to those axes: then

$$OE^2 : OZ^2 :: OR : RN,$$

and

$$OZ^2 : OE^2 :: OR' : R'N'.$$

The co-ordinate axes coinciding with OE and OZ, let x', y' be co-ordinates of P′, and x, y those of any point in the normal P′N; also representing OE and OZ, respectively, by t and c: then the equation for a normal to an ellipse will be [(d') Art. 92.]

$$t^2y'x' - t^2y'x = c^2x'y' - c^2x'y:$$

but, at the point N, where $y=0$, and $x=$ ON, the equation becomes

$$t^2(x'-x) = c^2x'; \text{ whence } x'-x = \frac{c^2}{t^2}x':$$

that is, $\quad \text{NR} = \frac{c^2}{t^2}x'$, or $t^2 : c^2 :: x' \,(=\text{OR}) : \text{NR}. \qquad (a)$

In the equation for the normal making $x=0$, in which case $-y=$ ON′, that equation becomes

$$t^2y'x' = c^2x'y' - c^2x'y,$$

or

$$\frac{t^2}{c^2}y' = y' - y;$$

whence we have

$$c^2 : t^2 :: y' \,(=\text{OR}') : y'-y \,(=\text{OR}' + \text{ON}') = \text{R}'\text{N}'.$$

The equation for a normal to an hyperbola is [(e') Art. 92.]

$$t^2 y'x - t^2 y'x' = c^2 x'y' - c^2 x'y,$$

from which, when $y=0$ we get

$$x - x' \; (=\text{NR}) = \frac{c^2}{t^2} x' : \qquad (b)$$

or, on making $x=0$,

$$y - y' \; (=\text{R}'\text{N}') = \frac{t^2}{c^2} y'.$$

111. DEF. 19. The part of either axis of an ellipse or hyperbola between the points at which the axis is intersected by a normal drawn from any point in the curve, and by an ordinate to the axis, drawn from the same point, is called a subnormal.

112. COR. 1. A tangent and a normal being drawn from any point in an ellipse or hyperbola, to meet the transverse axis of the curve in points, as T and N, we have

$\text{ON} \,.\, \text{OT} = \text{OE}^2 \mp \text{OZ}^2$ (=(Art. 69.) the square of the excentricity.)

[Note. The upper sign is for the ellipse, and the lower for the hyperbola.]

For, t and c being put for the semi-transverse and semi-conjugate axes, respectively, of either curve, and e for the excentricity,

$$\text{ON} = x' \mp \frac{c^2 x'}{t^2}, \text{ or} = \frac{t^2 \mp c^2}{t^2} x',$$

and $$\text{OT (Art. 93.)} = \frac{t^2}{x'};$$

therefore $$\text{ON} \,.\, \text{OT} = t^2 \mp c^2 = e^2.$$

113. COR. 2. A normal being drawn from any point P′ in an ellipse or hyperbola to cut the transverse and conjugate axes of the curve in points, as N and N′;

$$\text{OE}^2 \,.\, \text{P}'\text{N} = \text{OZ}^2 \,.\, \text{P}'\text{N}'.$$

For $$\frac{t^2}{c^2} \,.\, \text{OR}' = \text{R}'\text{N}', \text{ or } \frac{t^2}{c^2} = \frac{\text{R}'\text{N}'}{\text{OR}'};$$

but, from the similarity of the triangles P′R′N′, NON′,

$$\frac{\text{R}'\text{N}'}{\text{OR}'} = \frac{\text{P}'\text{N}'}{\text{P}'\text{N}};$$

therefore $$\frac{t^2}{c^2} = \frac{\text{P}'\text{N}'}{\text{P}'\text{N}}, \text{ or EO}^2 \,.\, \text{P}'\text{N} = \text{OZ}^2 \,.\, \text{P}'\text{N}'.$$

114. COR. 3. A tangent and normal being drawn as in the first and second corollaries;

$$\text{NR} \,.\, \text{OT} = \text{OZ}^2, \text{ and } \text{N}'\text{R}' \,.\, \text{OT}' = \text{OE}^2.$$

For $\quad NR = \frac{c^2}{t^2} OR$, and $OT = \frac{t^2}{OR}$;

therefore $\quad NR \cdot OT = c^2 \; (= OZ^2)$.

Also $\quad N'R' = \frac{t^2}{c^2} OR'$, and $OT' = \frac{c^2}{OR'}$;

therefore $\quad N'R' \cdot OT' = t^2 \; (= OE^2)$.

115. COR. 4. A tangent and a normal being drawn as before, and a perpendicular OV let fall on the former from the centre of the curve;

$$P'N' \cdot VO = t^2,$$

and $$P'N \cdot VO = c^2:$$

For, from the similarity of the triangles $P'N'R'$, TOV,

$$P'N' : P'R' :: OT : OV;$$

whence $$P'N' \cdot OV = P'R' \cdot OT$$
$$= OR \cdot OT = (\text{Art. } 101.)\; t^2.$$

Also, from the similarity of the triangles $P'NR$, TOV,

$$P'N : NR :: OT : OV;$$

whence $$P'N \cdot OV = NR \cdot OT = (\text{Art. } 114.)\; c^2.$$

PROPOSITION XXIII.

116. If a normal be drawn from any point in a parabola to cut the axis, and if an ordinate to the axis be drawn from the same point, the distance on the axis between the points in which it is cut by the normal and ordinate is equal to half the parameter of the axis.

That is, if GEH (second figure to Prop. XXII.) be considered as part of a parabola, RN is equal to half the parameter of the axis EX.

For p representing the parameter of the axis, and x', y' the co-ordinates of P'; also x, y the co-ordinates of any point in $P'N$, the origin being at the vertex E, the equation of a normal at P' is [(f) Art. 92.]

$$2y'x - 2y'x' = py' - py:$$

hence, at the point N where y vanishes, and x becomes EN, we have

$$x - x' \; (= NR) = \tfrac{1}{2}p.$$

PROPOSITION XXIV.

117. If lines be drawn from the foci of an ellipse or hyperbola to meet at any point in the curve; also, if a line be drawn from the focus of a parabola to any point in the curve, and a diameter be drawn through that point; these lines shall make equal angles with a tangent to the curve at the same point.

Fig. 1.

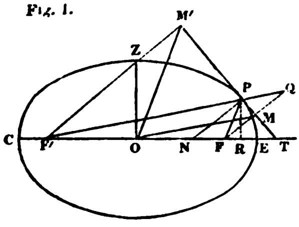
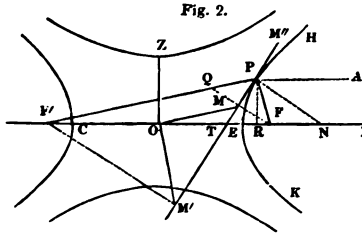


Fig. 2.



For an ellipse or hyperbola, let PT be a tangent meeting the transverse axis EC, produced, if necessary, in T, and let F, F′ be the foci; join F, P and F′, P, and imagine FM, F′M′ to fall perpendicularly on PT, produced, if necessary; then

the angle FPM = F′PM′.

The ordinate PR being let fall on CE, if the semi-axes of the curve be represented by t and c; also, O being the centre, if the excentricity OF be represented by e, and OR by x, we have for the ellipse,

$$\text{F}'\text{P} = \frac{t^2+ex}{t}, \text{ and } \text{FP} = \frac{t^2-ex}{t} \qquad [(a) \text{ Art. } 82.];$$

also, since (Art. 93.) $\text{OT} = \frac{t^2}{x}$, and OF or $\text{OF}' = e$,

$$\text{F}'\text{T} = \frac{t^2}{x} + e, \text{ or } \frac{t^2+ex}{x},$$

and $$\text{FT} = \frac{t^2}{x} - e, \text{ or } \frac{t^2-ex}{x}.$$

Now $$\frac{t^2+ex}{t} \sin. \text{F}'\text{PM}' = \frac{t^2+ex}{x} \sin. \text{FTP},$$

each of these being equal to F′M′;

also $$\frac{t^2-ex}{t} \sin. \text{FPM} = \frac{t^2-ex}{x} \sin. \text{FTP},$$

each of these being equal to FM.

Therefore $\sin. \text{F}'\text{PM}' = \frac{t}{x} \sin. \text{FTP}.$

and $\sin. \text{FPM} = \frac{t}{x} \sin. \text{FTP}:$

consequently the angles F′PM′ and FPM are equal to one another.

And, in like manner, using [(b) Art. 82.] $\frac{ex+t^2}{t}$ for F′P, and $\frac{ex-t^2}{t}$ for FP, may the proposition be proved for the hyperbola.

Also, if a normal PN be drawn from P, the angles FPN and F′PN in the ellipse, likewise the angle FPN, and the supplement of F′PN in the hyperbola, will be equal to one another.

For a parabola, if the branch HEK (fig. 2.) be considered as part of such a curve, F the focus, and PA a diameter passing through P, we shall have (Art. 107.) FP=FT; and FTP =FPT.

But the diameter PA being parallel to axis EX,

the angle FTP=APM″:

consequently FPT=APM″.

Also, PN being a normal, or perpendicular to PT,

the angle FPN=NPA.

118. COR. If a line be drawn from the centre of an ellipse or hyperbola to the point in which a line drawn from either focus perpendicularly on a tangent to the curve meets that tangent, it will be equal to the semi-transverse axis.

Let FM meet F′P, either of them being produced, if necessary, in Q; then the triangles FPM and QPM will be equal to one another. For the angles at M are right angles, the angle QPM=F′PM′=MPF, and PM is common. It follows that PQ=PF, and that F′Q=F′P+PF=(Art. 83.) 2EO, or F′Q=2t (t representing the semi-transverse axis).

Again, if O, M be joined by a straight line, OM will be parallel to F′Q, and equal to OE or t. For F′O=OF, and FM =MQ; therefore F′F and FQ are cut proportionally in O and M; and

$$\text{F}'\text{F} : \text{FO} :: \text{F}'\text{Q} : \text{OM};$$

or, e being the excentricity,

$$2e : e :: 2t : \text{OM} \ (=t).$$

In like manner OM′=t.

Since OE, OC, OM, OM′ are all equal to one another, a circle, described upon the transverse axis of an ellipse or hyperbola as a diameter, will pass through the intersections of any tangent with the perpendiculars let fall upon it from the two foci.

Scholium. A tangent to an ellipse, an hyperbola, or a parabola, may be drawn through any given point in the following manner.

1. For an ellipse, CE, fig. 1., being the transverse axis, and F and F′ the foci, if the given point be in the periphery, as at P, draw FP and F′P, and produce one of these lines, as FP, towards H: then, bisecting the angle F′PH by a line, as P′PT, drawn through P; P′T making equal angles with PF and PF′, will, by the proposition, be the required tangent. If the given point, as P′, be not in the periphery: with P′ as a centre, and a radius equal to the distance from thence to one of the foci, as F′, describe an arc, as F′H; and from F the other focus as a centre, with a radius equal to CE, describe an arc intersecting the former arc, as at H. Join F, H by a straight line intersecting the curve in some point, as P: then a line drawn through P′ and P will touch the curve in the latter point. For the lines being drawn as the figure, P′F′=P′H by construction, PF′=PH, since FP+PH and FP+PF′ are each equal to the transverse axis, and P′P is common to the triangles P′PH and P′PF′; therefore the angle P′PF=P′PH, or F′PT=TPH: consequently, by the proposition, P′PT is a tangent. If from P′ as a centre, with a radius equal to P′F, an arc be described, and from F′ as a centre with a radius equal to CE, an arc be described to cut the former arc in H′: then, on joining F′, H′ by a line cutting the curve in P″, a line drawn through P′P″ will be a tangent to the ellipse at P″.

Fig. 1.

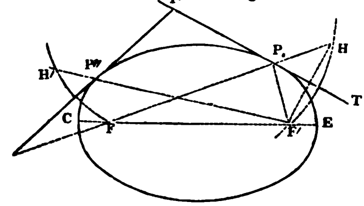
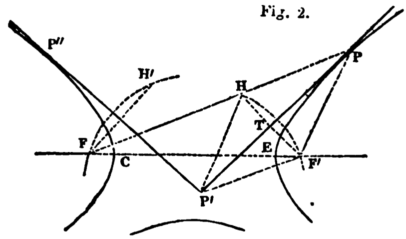


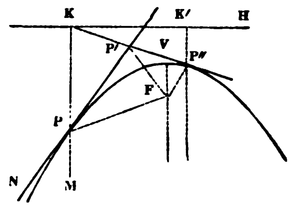
Fig. 2.



2. A construction precisely similar to that which has been given may be used (fig. 2.) for an hyperbola.

3. For a parabola. Let F, fig. 3., be the focus, V the

vertex, and KH the directrix: then, if the given point P be in the curve line, join P, F, and draw KP parallel to the axis VF; the line PP′, drawn to bisect the angle KPF, will, by the proposition, be the required tangent. If the given point P′ be not in the curve line, join P′, F, and with P′ as a centre, and a radius equal to P′F, describe an arc cutting the directrix in K or K′: then, drawing KP and K′P″ parallel to VF, the points P and P″ will be the points of contact, and the lines P′P, P′P″ will, by the proposition, be the required tangents. For the lines being drawn as in the figure, KP = PF (Art. 88. *a.*) KP′ = P′F by construction, and PP′ is common to the triangles PKP′, PFP′; therefore the angle FPP′ = KPP′ or MPN. In like manner it may be shown that the angle FP″P′ = P′P″K′.



PROPOSITION XXV.

119. To find the value of a perpendicular let fall from one of the foci of an ellipse or hyperbola, or from the focus of a parabola, upon a tangent drawn through any point of the curve.

The angle FTP (figs. to Prop. XXIV.) is equal to NPR, since the triangle NPT is right-angled at P, and PR is perpendicular to NT; therefore

$$\sin.^2 \text{FTP} = \sin.^2 \text{NPR}.$$

Now $\sin.^2 \text{NPR} = \dfrac{\text{NR}^2}{\text{NP}^2}$, in which expression $\text{NR}^2 = \dfrac{c^4}{t^4}x^2$ ((*a*) Art. 110.)

$$\text{and NP}^2\ (= \text{PR}^2 + \text{RN}^2) = \frac{c^2}{t^2}(t^2 - x^2) + \frac{c^4}{t^4}x^2;$$

$$= \frac{c^2}{t^2}\left\{t^2 - x^2 + \frac{c^2}{t^2}x^2\right\};$$

$$= (\text{since } t^2 - c^2 = e^2 \text{ (Art. 69.)})\ \frac{c^2}{t^4}(t^4 - e^2x^2);$$

therefore $\sin.^2 \text{NPR}$, or $\sin.^2 \text{FTP} = \dfrac{c^2x^2}{t^4 - e^2x^2}$.

But in the ellipse (Art. 93.) $\text{FT} = \dfrac{t^2}{x} - e$, or $\dfrac{t^2 - ex}{x}$,

and $\text{F}'\text{T} = \dfrac{t^2}{x} + e$, or $\dfrac{t^2 + ex}{x}$.

Multiplying the square of one of these values into the values of sin.^{2} FTP, we have

$$\frac{c^2(t^2 \pm ex)^2}{t^4 - e^2x^2}, \text{ or } c^2\frac{t^2 \pm ex}{t^2 \mp ex}, \text{ or again, [(a) Art. 82.], } c^2\frac{\text{FP}}{\text{PF}'},$$

for the value of FM^2 or $\text{F}'\text{M}'^2$.

In like manner, for FT using $\frac{ex-t^2}{x}$, for F′T, $\frac{ex+t^2}{x}$ and for PR^2, $\frac{c^2}{t^2}(x^2-t^2)$ it will be found that, in an hyperbola,

$$\text{FM}^2 \text{ or } \text{F}'\text{M}'^2 = c^2\frac{ex \pm t^2}{ex \mp t^2}.$$

Scholium. Putting r for PF, we have (Art. 83.) for F′P in an ellipse, $2t-r$, and in an hyperbola, $2t+r$. Thus the square of a perpendicular let fall from either of the foci upon a tangent may be represented by

$$c^2\frac{r}{2t \mp r}:$$

the upper sign being used for an ellipse, and the lower for an hyperbola.

For a parabola, since $\frac{\text{PR}}{\text{RT}}$ denotes the tangent of the angle FTP or (Art. 107.) FPT; on substituting the values of the squares of those lines, we have (Arts. 72. and 104.)

$$\frac{px}{4x^2}, \text{ or } \frac{p}{4x}, = \tan.^2 \text{FPT}; \text{ and } \frac{4x}{p} = \text{cotan.}^2 \text{FPT}:$$

hence (Pl. Trigon. Art. 26. d) $1+\frac{4x}{p}$ or $\frac{p+4x}{p} = \text{cosec.}^2 \text{FPT}$,

and $$\frac{p}{p+4x} = \sin.^2 \text{FPT}.$$

Now $\text{FP} = \frac{p}{4}+x$, or $\frac{p+4x}{4}$ [(c) Art. 82.], and $\text{FP}^2 = \frac{(p+4x)^2}{16}$

Multiplying, we get

$$\frac{p(p+4x)}{16} = \text{FM}^2$$

(the square of a perpendicular let fall from the focus upon the tangent).

Putting r for $\frac{p+4x}{4}$ (= FP) we have

$$r\frac{p}{4} = \text{FM}^2.$$

120. COR. The rectangle or product of the perpendiculars let fall from the foci upon a tangent to an ellipse or hyperbola is equivalent to the square of the semi-conjugate axis. That is $\text{FM}.\text{F}'\text{M}' = \text{ZO}^2$.

From the proposition we have, for an ellipse,

$$\text{FM} = c\sqrt{\frac{t^2+ex}{t^2-ex}} \text{ and } \text{F}'\text{M}' = c\sqrt{\frac{t^2-ex}{t^2+ex}}:$$

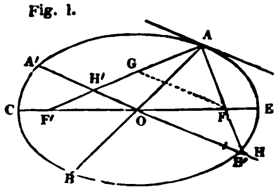
consequently, $\text{FM}.\text{F}'\text{M}' = c^2$.

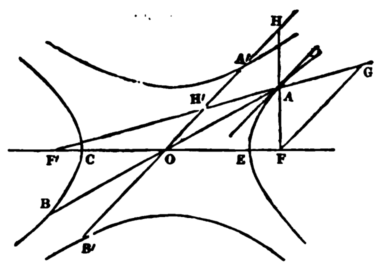
A like result will be obtained for an hyperbola.

PROPOSITION XXVI.

121. If from any point in an ellipse or hyperbola two lines be drawn to the foci, one of them being produced if necessary; the segment of either line, intercepted between the point and a diameter which is conjugate to one passing through the same point, is equal to the semi-transverse axis of the curve.

Fig. 1.





Let F and F′ be the foci, and A any point in the ellipse or hyperbola; let also AB be a diameter passing through A, and A′B′ be conjugate to it; and again, let AF and AF′, either of them produced if necessary, meet A′B′ in H and H′; then

$$\text{AH or AH}' = \text{OE (the semi-transverse axis).}$$

Draw FG parallel to A′B′; this line will consequently be parallel to a tangent at A; whence the angles AFG and AGF, being equal to the angles which AF and AF′ make at A with the tangent, will be equal to one another, and we shall have

$$\text{AG} = \text{AF}.$$

But, since FF′ is bisected in O, F′G will be bisected in H′; consequently

$$\text{AH}' = \tfrac{1}{2}(\text{AF}' + \text{AF}) = \text{(Art. 83.)}\ \tfrac{1}{2}\,\text{CE}:$$

that is, $\text{AH}' = \text{OE}$.

Also, F G being parallel to A′B′, A H and A H′ are cut proportionally in F and G (Euc. 2. VI.); but A G = A F; therefore

$$AH = AH', = OE.$$

PROPOSITION XXVII.

122. At any point in an ellipse or hyperbola let a tangent be drawn to meet any diameter produced, and from the point of contact let an ordinate to that diameter be drawn, also from the centre and the two extremities of the diameter let lines be drawn parallel to the ordinate; then, if these lines be taken in order, the first will be to the second as the third is to the fourth.

Fig. 1.

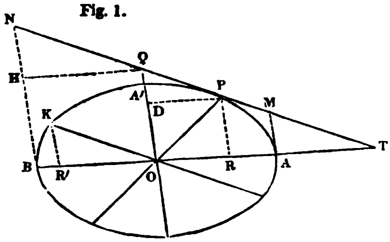
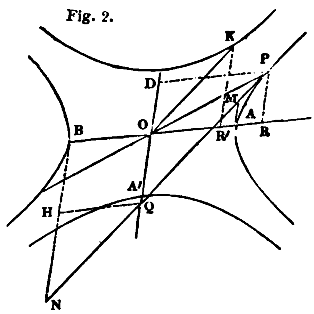


Fig. 2.



Let A B be any diameter of an ellipse or hyperbola, N P T a tangent at any point P; let P R be the ordinate, and A M, O Q, B N, be lines parallel to it, so that O A′ may be a semi-diameter conjugate to A B and A M, B N may be tangents to the curve at A and B; then

$$AM : PR :: OQ : BN.$$

Let $OA = a$, $OR = x$; then, the tangent meeting the diameter A B in T,

$$OT \text{ (Art. 93.)} = \frac{a^2}{x};$$

consequently, for an ellipse,

$$AT = \frac{a^2 - ax}{x},\ TR = \frac{a^2 - x^2}{x} \text{ and } TB = \frac{a^2 + ax}{x}.$$

Hence $$TA : TR :: \frac{a^2 - ax}{x} : \frac{a^2 - x^2}{x},$$

or $\qquad \text{TA} : \text{TR} :: a : a+x;$

and $\qquad \text{TO} : \text{TB} :: \dfrac{a^2}{x} : \dfrac{a^2+ax}{x},$

or $\qquad :: a : a+x.$

Thus $\qquad \text{TA} . \text{TR} :: \text{TO} : \text{TB}:$

consequently, from the similarity of the triangles TAM, TRP, &c. $\qquad \text{AM} : \text{PR} :: \text{OQ} : \text{BN}.$

In like manner may the proposition be proved for an hyperbola.

123. COR. 1. If a tangent be drawn at any point P in an ellipse or hyperbola, and other tangents be drawn from the extremities A and B of any diameter, to cut the first tangent, as at M and N; the rectangle of these two tangents is equivalent to the square of the semi-diameter which is conjugate to that diameter; that is,

$$\text{AM}.\text{BN} = \text{OA}'^2.$$

Let the semi-diameters OA, OA′ be represented by a and b; then, from the similarity of the triangles as in the proposition,

$$\text{TR} : \text{RP} :: \text{TA} : \text{AM},$$

or, for the ellipse,

$$\frac{a^2-x^2}{x} : \frac{b}{a}(a^2-x^2)^{\frac{1}{2}} :: \frac{a^2-ax}{x} : \text{AM}\left(=b\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\right);$$

also, $\qquad \text{TR} : \text{RP} :: \text{TB} : \text{BN},$

or $\qquad \dfrac{a^2-x^2}{x} : \dfrac{b}{a}(a^2 x^2)^{\frac{1}{2}} :: \dfrac{a^2+ax}{x} : \text{BN}\left(=b\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}\right):$

consequently $\qquad \text{AM}.\text{BN} = b^2\ (=\text{OA}'^2).$

The like proof will serve for an hyperbola.

124. COR. 2. If a tangent be drawn at any point P in an ellipse or hyperbola, and others be drawn from the extremities of any diameter, as AB, to cut the first tangent as before; the rectangle of the segments of that tangent, between the point of contact and the two intersections, is equivalent to the square of a semi-diameter which is conjugate to the one passing through the point of contact;

that is, OK being a semi-diameter conjugate to OP,

$$\text{PM}.\ \text{PN} = \text{OK}^2.$$

Imagine KR′ to be an ordinate to AB at the point K, so that KR′ is parallel to BN, OA′, &c.; also imagine QH and PD to be drawn parallel to AB: then, from the similarity of the triangles,

$$\text{OR}'^2 : \text{OK}^2 :: \text{QH}^2\ (=\text{OB}^2) : \text{QN}^2\ \left(=\frac{\text{OK}^2}{\text{OR}'^2}.\text{OB}^2\right);$$

also $\text{OR}'^2 : \text{OK}^2 :: \text{PD}^2\ (=\text{OR}^2) : \text{PQ}^2\ \left(=\frac{\text{OK}^2}{\text{OR}'^2}\ \text{OR}^2\right).$

But (Art. 77.) $\text{OR}^2 = \text{OB}^2 - \text{OR}'^2;$

therefore $\text{PQ}^2 = \frac{\text{OK}^2}{\text{OR}'^2}\ (\text{OB}^2 - \text{OR}'^2).$

Subtracting, we get

$$\text{QN}^2 - \text{PQ}^2 = \text{OK}^2.$$

But, since AB is bisected in O, and AM, OQ, BN are parallel to one another, MN is bisected in Q;

therefore (Euc. 5. II.) $\text{QN}^2 - \text{PQ}^2 = \text{MP}.\text{PN};$

and consequently $\text{MP}.\text{PN} = \text{OK}^2.$

PROPOSITION XXVIII.

125. When two chords intersect one another in an ellipse or hyperbola, the rectangle contained by the segments of one will be, to the rectangle contained by the segments of the other, as the square of the semi-diameter which is parallel to the first is to the square of that which is parallel to the second. Also, in a parabola, the rectangles of the segments of two intersecting chords are to one another as the squares of the parameters of the diameters to which they are double ordinates.

Fig. 1.

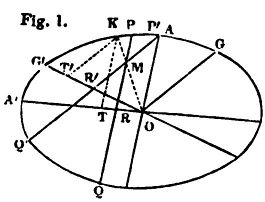
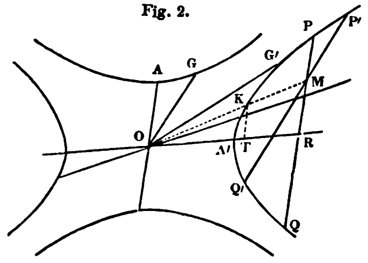


Fig. 2.



In both figures, let PQ, P′Q′ be two chords intersecting

one another in M; also let OA be a semi-diameter parallel to PQ, and OG one parallel to P′Q′: then

$$\text{PM}.\text{MQ} : \text{P}'\text{M}.\text{MQ}' :: \text{OA}^2 : \text{OG}^2.$$

Let OA′, OG′ be semi-diameters conjugate to OA and OG respectively; the chords PQ, P′Q′ being consequently bisected by those semi-diameters, and the former in the point R. Through M draw the semi-diameter OK, and draw KT parallel to AO.

Let $\text{OA}' = a$, $\text{OA} = b$, $\text{OR} = x$, $\text{RP} = y$;
also $\text{OT} = x'$, $\text{TK} = y'$.

Then ((*a*) Art. 56.) in an ellipse,

$$\frac{b^2}{a^2}(a^2 - x^2) = y^2, \text{ or } b^2 - \frac{b^2}{a^2} x^2 = y^2, \tag{1}$$

and

$$b^2 - \frac{b^2}{a^2} x'^2 = y'^2. \tag{2}$$

By the similarity of the triangles OKT, OMR,

$$\frac{\text{OM}}{\text{OK}} = \frac{\text{OR}}{\text{OT}} \left(= \frac{x}{x'}\right) = \frac{\text{MR}}{\text{KT}};$$

and, for the point M, each of these ratios is constant; let either of them be represented by m. Then,

from the equation $\frac{\text{MR}}{\text{KT}} = m$ we get $\text{MR}^2 = m^2 \text{KT}^2$

$$= m^2 y'^2,$$

and from the equation $\frac{\text{OR}}{\text{OT}} = m$ we have $\frac{x^2}{x'^2} = m^2$.

Now, in the equation (2) multiplying b^2, in the first term, and y'^2 by m^2, and the second term of the first member by $\frac{x^2}{x'^2}$ (the equivalent of m^2) that equation becomes

$$b^2 m^2 - \frac{b^2}{a^2} x^2 = m^2 y'^2. \tag{3}$$

And subtracting this last equation from (1) we have

$$b^2 - b^2 m^2 = y^2 - m^2 y'^2.$$

But the second member of this equation is $\text{RP}^2 - \text{MR}^2$; which, since PQ is bisected in R is (Euc. 5. II.) equivalent to PM.MQ.

Thus $\text{PM}.\text{MQ} = b^2 (1 - m^2)$.

In like manner it may be shown, representing OG by b', and drawing an ordinate KT′ from K to the diameter OG′, that

$$\text{P}'\text{M}.\text{MQ}' = b'^2 (1 - m^2);$$

Consequently

$$\text{PM.MQ} : \text{P}'\text{M.MQ} :: b^2 : b'^2,$$

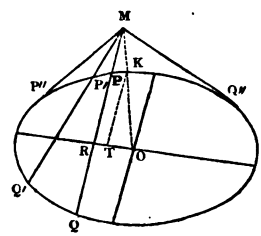
or the rectangles of the segments of the chords are to one another as the squares of the semi-diameters which are parallel to the chords.

a. If the point M were on the exterior of the ellipse, on subtracting (1) from (3) we should have

$$\text{MR}^2 - \text{RP}^2 \;(= \text{PM.MQ}) = b^2\,(m^2-1);$$

and in like manner,

$$\text{P}'\text{M.MQ}' = b'^2\,(m^2-1):$$



Consequently the rectangles of the produced chords, and the parts beyond the ellipse, are to one another as the squares of the semi-diameters which are parallel to such chords.

Again, if the point M were on the exterior of the ellipse, and a line MPQ were drawn through, while another line MP$''$ were to touch the ellipse; we should, as before, b and b' representing the semi-diameters parallel to MQ and MP$''$, get

$$\text{PM.MQ} = b^2\,(m^2-1):$$

while, after multiplying the equation corresponding to (2) above by m^2 or its equivalent, we should have

$$\text{MP}''^2 = b'^2\,(m^2-1).$$

Therefore,

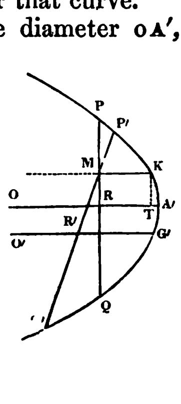
$$\text{PM.MQ} : \text{MP}''^2 :: b^2 : b'^2.$$

And it is easy to perceive that, if two lines MP$''$, MQ$''$ were drawn from one point M to touch the ellipse, the squares of those tangents would be, to one another, as the squares of the semi-diameters of the ellipse, which are parallel to them.

The demonstration of the proposition, in all its cases, would be the same for an hyperbola, on employing, instead of (1) and (2) the corresponding equations for that curve.

b. With respect to a parabola: since the diameter OA$'$, which bisects PQ in R, is parallel to MK, the diameter drawn through M, we have

$$\text{MR} = \text{KT}.$$



But A$'$R and A$'$T being represented by x and x', also RP by y, and TK or MR by y'; and again, p being the parameter of the diameter OA$'$, we have (Art. 72.)

$$px = y^2,$$

and $$px' = y'^2:$$

therefore,

$$p\,(x-x')=y^2-y'^2.$$

But $x-x'$ is constant for the point M; let this constant be represented by m; then

$$pm=y^2-y'^2\ (=\text{PM}.\text{MQ}).$$

In like manner, p' being the parameter of the diameter O′G′ to which P′Q′ is a double ordinate,

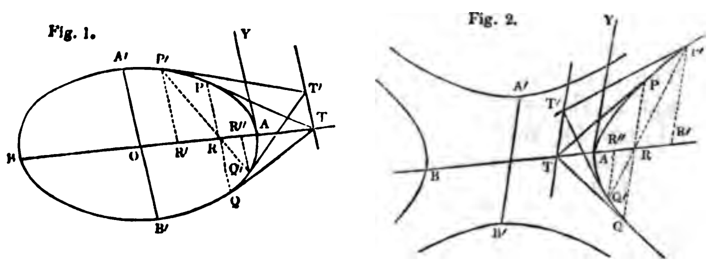
$$p'm=\text{P}'\text{M}.\text{MQ}'.$$

Thus the rectangles of the segments of the chords are to one another as the parameters of the diameters of which the chords are double ordinates; or (Art. 108.) as the distances of the focus from the vertices of those diameters.

By similar processes it may be proved that, in a parabola, the rectangles of the segments of lines drawn from a point on the exterior, or the squares of the tangents drawn from a point on the exterior, are to one another as the squares of the parameters of diameters parallel to the lines.

PROPOSITION XXIX.

126. If chords are drawn in an ellipse, hyperbola or parabola, to intersect one another on any diameter, and tangents are drawn from the extremities of each to intersect one another, the points of intersection for every pair of tangents will be in one straight line parallel to a tangent at the vertex of that diameter.



Let PQ in both figures be a chord intersecting the diameter AB, of which it is a double ordinate, in R, and let the tangents at P and Q meet (Art. 97.) in T. Let P′Q′ be any other chord passing through R, and let P′T′, Q′T′ be tangents at P′ and Q′, meeting in some point T′; then T′ will be in a straight line, drawn through T parallel to A′B′, which is conjugate to AB, or parallel to a tangent to the curve at A. Let OA $(=a)$,

OA′ ($=b$) be considered as the co-ordinate axes; let x' and y' ($=$OR′ and R′P′) represent the co-ordinates of P′, and let x, y be the co-ordinates of any point in P′T′.

Then, for an ellipse, the equation for a tangent as P′T′, is [(a') Art. 90.],

$$a^2yy' + b^2xx' = a^2b^2,$$

and for the tangent Q′T′,

$$-a^2y_,y'' + b^2x_,x'' = a^2b^2,$$

x'' and $-y''$ ($=$OR″ and R″Q′) being the co-ordinates of Q′, while $x_,$ and $y_,$ are those of any point in Q′T′.

But, at T′ the point of intersection, $x_,$, $y_,$ are respectively equal to x and y; therefore, subtracting the first equation from the second,

$$-a^2y''(y''+y') + b^2x(x''-x') = 0;$$

whence

$$\frac{x''-x'}{y''+y'} = \frac{a^2y}{b^2x}.$$

Now, from the similarity of the triangles RR′P′, RR″Q′, we have RR′ : RR″ :: R′P′ : R″Q′;

whence, by conversion, RR′ : R′R″ :: R′P′ : R′P′+R″Q′,

or

$$\frac{\mathrm{RR}'}{\mathrm{R'P'}} = \frac{\mathrm{R'R''}}{\mathrm{R'P'}+\mathrm{R''Q'}}:$$

But the second member of this equation is equivalent to $\frac{x''-x'}{y''+y'}$; therefore putting OR$-$OR′, or OR$-x'$, for RR′, we have

$$\frac{\mathrm{OR}-x'}{y'} = \frac{a^2y}{b^2x};$$

whence $\mathrm{OR}.b^2x = a^2y'y + b^2x'x \ (=a^2b^2)$:

therefore $\mathrm{OR}.x = a^2$, and $x = \dfrac{a^2}{\mathrm{OR}}$.

Thus x, or OT, is constant for all chords passing through R, and it follows that the intersection T′ of the pairs of tangents to the ellipse, at the extremities of such chords, is in a line TT′ passing through T parallel to A′B′ or AY.

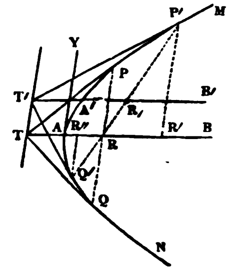
If AB were the transverse axis, and R the focus of the ellipse, it is evident (Art. 88.) that all the points T, T′, &c. would be in the directrix.

In like manner may the proposition be demonstrated for an hyperbola, in which the equation of a tangent is [(b') Art. 90.]

$$b^2xx' - a^2yy' = a^2b^2.$$

For a parabola, let MAN be part of the curve; then the equation for a tangent as P′T′ being [(c'') Art. 90.]

$$2yy'=px+px',$$



in which, with reference to the diameter AB, x' and y' are the co-ordinates of P′, the point of contact, x and y the co-ordinates of any point in P′T′, and p the parameter of AB; also the equation for Q′T′ being

$$-2y_{,}y''=px_{,}+px'',$$

by subtraction we have, for the point T′ at which $x_{,}=x$ and $y_{,}=y$,

$$2(y'+y'')y=p(x'-x'');$$

whence $\frac{2y}{p}=\frac{x'-x''}{y'+y''}$ $\left(=\text{, by similarity of triangles, } \frac{\text{R}'\text{R}}{\text{P}'\text{R}'}\right)$.

Then, for R′R putting AR′ − AR, or x' − AR, we have

$$\frac{2y}{p}=\frac{x'-\text{AR}}{y'}, \text{ or } 2yy'=px'-p\,.\,\text{AR}:$$

putting for $2yy'$ its equivalent $px+px'$ above, the equation becomes

$$x=-\text{AR}.$$

Thus for the point T′ of intersection, x is constant when the chords pass through R; and therefore all the pairs of tangents intersect one another in a line passing through T parallel to AY, or to any of the ordinates of AB.

If AB were the axis and R the focus of the parabola, it is evident (Art. 88.) that all the pairs of tangents would meet on the directrix.

127. COR. In a parabola, the two tangents drawn from the extremities of every chord passing through the focus will meet on the directrix at right angles to one another.

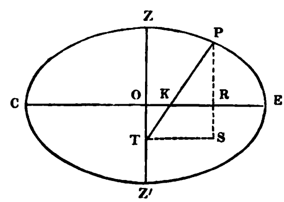
It has been already proved that tangents so drawn intersect one another on the directrix. Let TT′ be the directrix, which will then be at right angles to AB, the axis; also, let P′T′, Q′T′ be two such tangents, and A′B′ a diameter drawn through T′; then, since tangents drawn from the extremities of a double ordinate to that diameter meet on that diameter produced, P′Q′ is a double ordinate to A′B′, and is consequently bisected in R,. Now A′ being the vertex of that diameter, and R the focus of the parabola, T′A′, A′R,, and A′R are each equal (Art. 72.) to one quarter of the parameter of A′B′, while R,P′ and R,Q′ are each equal to half that para-

meter; therefore $R_{,}P'$, $R_{,}Q'$, and $A'T'$ are equal to one another, and the points P', T', Q' are in the semi-circumference of a circle described on $P'Q'$ as a diameter: it follows that $P'T'Q'$ is a right angle.

PROPOSITION XXX.

128. **If a straight line equal to the semi-transverse axis of an ellipse be applied from any point in the periphery of such curve to the conjugate axis, the segment of this line between the curve and the transverse axis will be equal to the semi-conjugate axis.**

Let PT (= OE the semi-transverse axis) be the line so applied, and let it meet OE in K; then



PK = ZO (the semi-conjugate axis). Through P draw to the transverse axis, the ordinate PR, and produce it till it meets, in S, a line drawn through T parallel to OE. Then O being the origin of the co-ordinates, if OE be represented by t, OZ by c, and OR by x, we have [(a) Art. 56.]

$$PR = \frac{c}{t}\,(t^2 - x^2)^{\frac{1}{2}}.$$

But $PT = t$, and $TS\ (= OR) = x$; therefore $PS = (t^2 - x^2)^{\frac{1}{2}}$,

and $$PS : PR :: 1 : \frac{c}{t}.$$
$$:: t : c;$$

also, PS and PT being cut proportionally in R and K, we have

$$t : c\ (:: PS : PR) :: PT : PK;$$

but $PT = t$; therefore $PK = c\ (= OZ)$.

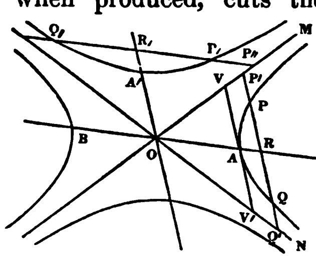
Scholium. Any number of points in the periphery of an ellipse may be found in the following manner:—Let CE be the transverse, and ZZ′ the conjugate axis; and let the line PT represent the thin edge of a ruler; then, if on that edge marks be made at P, K, and T, such that PK and PT shall be respectively equal to the semi-conjugate and semi-transverse axes of the ellipse; and if the ruler be moved so that K shall be on any part of CE, and T upon any part of ZZ′ produced if

necessary, the point P will (by the proposition) be always in the periphery of the ellipse. Any number of points, as P, may be thus found, and a curve line may be drawn through them.

An instrument by which ellipses may be described has been constructed conformably to the property demonstrated in this proposition. It consists of two bars of metal forming four arms at right angles to one another, as OC, OE, OZ, OZ′; each bar has in it a groove extending quite along it on its upper surface, and in each groove is a moveable pillar perforated at the top: through both perforations passes a bar, corresponding to the line PT, which lies in a plane parallel to that of the instrument, and is capable of being fixed by small screws at points corresponding to K and T; also at the end P is a socket carrying a pencil. Then, if this bar be screwed to the pillars so that PT may be equal to the semi-transverse, and PK to the semi-conjugate axis of the intended ellipse; on moving the bar, the pillars will slide in the grooves along CE and ZZ′, and the pencil will describe the periphery of the ellipse.

PROPOSITION XXXI.

129. If any chord be drawn in an hyperbola, and be produced till it meets the asymptotes, the rectangle of the segments between its intersections with the curve and the point in which it cuts either asymptote will be equal to the square of the semi-diameter which is conjugate to the diameter bisecting the chord.



Let PQ be a chord which, when produced, cuts the asymptotes OM, ON in P′, Q′, and let OR, cutting the curve in A, bisect PQ in R: let OA′ be a semi-diameter conjugate to OA, and let VV′ be drawn through A parallel to PQ or OA′; then PQ will be a double ordinate to the diameter BA and (Art. 38.) VV′ will be a tangent to the curve at A. It is required to prove that

$$QP'.P'P, \text{ or } PQ'.Q'Q, = OA'^2.$$

Put $a = OA$, $b = OA'$, $=$ (Art. 96.) AV, and $x = OR$; then [(a) Art. 57.]

$$PR^2, \text{ or } RQ^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

But RP′ or RQ′ being an ordinate to the asymptote, we have (Art. 59.)

$$RP'^2, \text{ or } RQ'^2 = \frac{b^2}{a^2} x^2;$$

therefore

$$RP'^2 - RP^2 \; (= QP'.P'P) = b^2 \; (= O'A^2 \text{ or } AV^2) \quad (a)$$

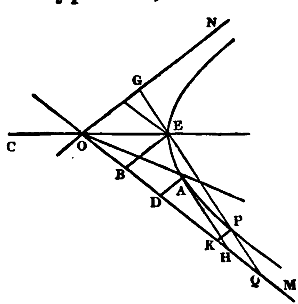
Using the equation $R_{,}P_{,}^2 = \frac{a^2}{b^2}(y^2 - b^2)$ ((b′) Art. 57.), and the corresponding equation $R_{,}P''^2 = \frac{a^2}{b^2} y^2$ (Art. 59.) for the asymptote, it may be proved in like manner that

$$R_{,}P''^2 - R_{,}P_{,}^2, \text{ or } Q_{,}P''.P_{,}P'', = a^2 \; (= OA^2). \quad (b)$$

PROPOSITION XXXII.

130. If through either extremity of the transverse axis of an hyperbola a chord be drawn, and it be produced to cut the asymptotes; also, if there be drawn a diameter to which that chord is a double ordinate; then lines drawn between the curve and one of the asymptotes parallel to the other, from the extremity of that diameter and the two extremities of the chord, will be in a geometrical progression.

Let OM, ON be asymptotes of an hyperbola, and EP a chord passing through E, one extremity of the transverse axis: let EP be produced to G and Q, and let OA be a semi-diameter to which EP is a double ordinate. Lastly, let EB, AD, and PK be drawn parallel to ON; it is required to prove that EB, AD, PK are in a geometrical progression.



Let AH be drawn parallel to EP; and consequently touching the hyperbola in A. Then by the similarity of the triangles we have

$$BE : EQ :: DA : AH,$$

and $$KP : PQ :: DA : AH;$$

therefore $$BE.KP : EQ.PQ :: DA^2 : AH^2.$$

But [(*a*) Art. 129.] $EQ.QP = AH^2$;

therefore $BE.KP = AD^2$,

or $BE : AD :: AD : KP$;

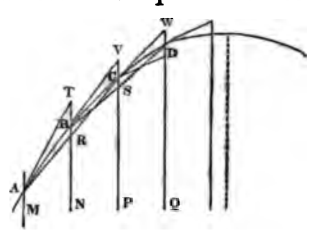
that is, BE, AD, KP are in a decreasing geometrical progression.

It follows also, since (Art. 62.) the rectangles OB.BE, OD.DA, OK.KP, &c. are equal to one another; that OB, OD, OK are reciprocally proportional to BE, AD, KP; and consequently are in an increasing geometrical progression.

PROPOSITION XXXIII.

131. In a parabola, if any number of diameters be drawn at equal distances from one another, and from the extremity of each diameter a tangent to the curve be drawn to meet the next diameter produced, the parts intercepted on all the diameters between the tangent and the curve will be equal to one another.

Let AM, BN, CP, DQ, &c. be diameters at equal distances from one another, and let AT, BV, CW, &c. be tangents to the curve at A, B, C, &c.; then



TB, VC, WD, &c. will be equal to one another.

Join A and C, B and D by lines cutting TN in R and VP in S; then since, by hypothesis, the distances AN, NP, PQ, &c. measured in any direction (perpendicular to the diameters for example) are equal to one another, we have

$$AR = RC, \quad BS = SD, \text{ \&c.};$$

therefore AC is an ordinate to the diameter BN, BD is an ordinate to VP, &c. Hence (Art. 38.) RC is parallel to BV, SD to VW, &c. Also (Art. 104.)

$$TB = BR, \quad VC = CS, \text{ \&c.}:$$

but (Euc. 34. I.) $BR = VC$, $CS = WD$, &c.;

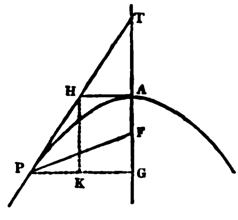
therefore TB, VC, WD are equal to one another.

PROPOSITION XXXIV.

132. If a tangent be drawn from any point in a parabola, and one from the extremity of the axis to

meet it; on drawing an ordinate to the axis from the point, the rectangle contained by the abscissa and a line drawn to the same point from the focus, is equal to the square of the intercepted part of the tangent.

Let P be any point in the curve, AG part of the axis, F the focus, and PG an ordinate to the axis; also let PT, AH be tangents drawn to the curve at P and A, the former meeting the axis at T; then



$$\text{AG.PF} = \text{PH}^2.$$

Draw HK parallel to AG; then by similar triangles, and because (Art. 104.) $\text{AT} = \frac{1}{2}\text{TG}$, $\text{AH} (= \text{KG}) = \frac{1}{2}\text{PG}$; therefore $\text{PK} = \frac{1}{2}\text{PG}$.

Putting p for the parameter of the axis, and x for AG or HK;

$$\tfrac{1}{4}px \text{ (Art. 72.) } = \text{PK}^2,$$

and $$\tfrac{1}{4}px + x^2, \text{ or } \frac{p+4x}{4}x = \text{PH}^2.$$

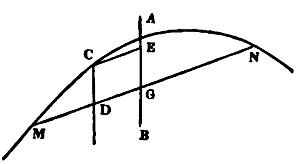
But [(c) Art. 82.] $\text{PF} = \frac{p+4x}{4}$; therefore $\text{AG.PF} = \frac{p+4x}{4}x$,

and consequently $\text{AG.PF} = \text{PH}^2$.

PROPOSITION XXXV.

133. If a double ordinate be drawn to a diameter of a parabola, and any other diameter be drawn, cutting that double ordinate and the curve, the rectangle of the segments of the ordinate is equal to that of the parameter of the axis and the segment of the latter diameter.

Let AB, CD be parts of any diameters, and MN a double ordinate to AB; then, p being the parameter of the same diameter,



$$p.\text{CD} = \text{MD.DN}.$$

Draw CE parallel to MN so that CE may be an ordinate to AB; put x for AE, and x' for AG; then (Art. 72.)

$$px = \text{CE}^2, \text{ and } px' = \text{MG}^2;$$

therefore $p\,(x'-x) = \text{MG}^2 - \text{CE}^2$;

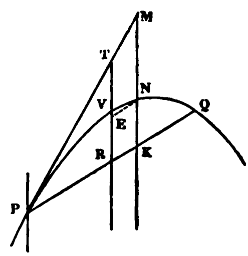
or, since $x'-x = \text{CD}$, and MN is bisected in G,

$$p\,.\,\text{CD} = \text{MD}\,.\,\text{DN}.$$

PROPOSITION XXXVI.

134. If from any point in a parabola a tangent and a chord be drawn; the segments of any diameter intercepted between the tangent, the curve line, and the chord, will be to one another in the same proportion as the segments of the chord intercepted between the diameter and the curve.

Let PM be the tangent, PQ the chord, and MK part of any diameter intersecting the chord in K, the tangent in M, and the curve in N: then



$$\text{MN} : \text{NK} :: \text{PK} : \text{KQ}.$$

Let TR be part of a diameter bisecting PQ in R, cutting the curve in V and the tangent in T, and draw NE parallel to PQ; thus both RQ and NE are ordinates to TR.

Put p for the parameter of the diameter TR, x for VE, and x' for VR, or (Art. 104.) TV: then (Art. 72.)

$$\sqrt{px} = \text{EN or RK, and } \sqrt{px'} = \text{PR or RQ};$$

therefore, adding and subtracting,

$$\sqrt{p}\,(\sqrt{x} + \sqrt{x'}) = \text{PK, and } \sqrt{p}\,(\sqrt{x'} - \sqrt{x}) = \text{KQ}.$$

Hence

$$\text{PK} : \text{KQ} :: \sqrt{x} + \sqrt{x'} : \sqrt{x'} - \sqrt{x}.$$

But (Art. 64. *a.*)

$$\text{PT}^2 : \text{PM}^2 :: \text{TV} : \text{MN}:$$

again, by similarity of triangles and Euc. 22. VI.

$$\text{PR}^2 : \text{PK}^2 :: \text{PT}^2 : \text{PM}^2;$$

therefore $\text{PR}^2 : \text{PK}^2 :: \text{TV} : \text{MN}$,

and $\text{MN} = \dfrac{\text{PK}^2}{\text{PR}^2}.\text{TV}$, or $= \dfrac{\text{PK}^2.x'}{p.x'}$, or again $= (\sqrt{x} + \sqrt{x'})^2$,

while NK (=ER) $= x' - x$

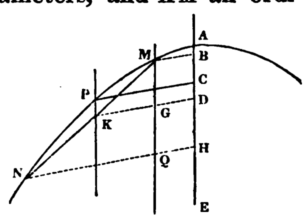
$$= (\sqrt{x} + \sqrt{x'})(\sqrt{x'} - \sqrt{x}):$$

therefore MN : NK :: $\sqrt{x} + \sqrt{x'} : \sqrt{x'} - \sqrt{x}$,
which, as above, is also the ratio of PK to KQ:
thus, finally, MN : NK :: PK : KQ.

PROPOSITION XXXVII.

135. If to any diameter of a parabola an ordinate be drawn, also from the vertex of the diameter, and from the intersection of the ordinate with the curve, ordinates be drawn to any other diameter, the rectangle contained by the parameter of the latter diameter and the abscissa of the former will be equal to the square of the difference between the two last ordinates.

Let PK, AE be parts of two diameters, and KM an ordinate to the former at any point K in it; let also MB, PC be ordinates to AE; then p being the parameter of AE,



$$p.\text{PK} = (\text{PC} - \text{MB})^2.$$

Produce MK to N, draw the lines NH, KD parallel to PC, and MQ parallel to AD, cutting KD in G; then (Art. 72.)

$$p.\text{AB} = \text{MB}^2,\ p.\text{AC} = \text{PC}^2, \text{ and } p.\text{AH} = \text{NH}^2.$$

Subtracting the first of these equations from the second,

$$p.\text{BC} = \text{PC}^2 - \text{MB}^2$$
$$= (\text{PC} + \text{MB})(\text{PC} - \text{MB}). \quad (a)$$

Again, subtracting the first from the third,

$$p.\text{BH} = \text{NH}^2 - \text{MB}^2 \quad (b)$$
$$= (\text{NH} + \text{MB})(\text{NH} - \text{MB}).$$

But MN being bisected in K, we have 2BD = BH, also

$$\text{NH} + \text{MB} = 2\text{KD or } 2\text{PC},$$

and $\text{NH} - \text{MB}\ (= \text{NQ or } 2\text{KG}) = 2(\text{PC} - \text{MB});$

therefore, substituting in (b) and dividing by 2,

$$p.\text{BD} = 2\text{PC}(\text{PC} - \text{MB}).$$

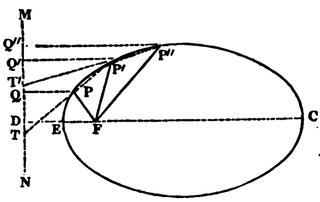
From this last equation subtracting (a) we have

$$p.\text{CD} = (\text{PC}-\text{MB}).(\text{PC}-\text{MB}),$$

or
$$p.\text{PK} = (\text{PC}-\text{MB})^2.$$

PROPOSITION XXXVIII.

136. Having a focus, with the lengths and positions of three radii vectores, and the angles contained between them; to find the axis of a conic section passing through the extremities of the radii.



Let FP, FP′, FP″ be the given radii, and imagine MN to be the directrix of the curve: join P′, P and P″, P′; and, producing the lines P′ P, P″ P′ indefinitely, imagine them to intersect the directrix in T and T′.

Now (Art. 89. a.)

$$\text{FP} : \text{FP}' :: \text{PT} : \text{P}'\text{T};$$

whence
$$\text{FP}'-\text{FP} : \text{FP}' :: \text{P}'\text{T}-\text{PT} : \text{P}'\text{T};$$

and, since the three first terms are known, the point T is determined.

In like manner, from the proportion

$$\text{FP}''-\text{FP}' : \text{FP}'' :: \text{P}''\text{T}'-\text{P}'\text{T}' : \text{P}''\text{T}',$$

the point T′ is determined: and thus the position of the directrix is found.

From the points P, P′, P″ let fall PQ, P′Q′, P″Q″ perpendicularly on the directrix; and, through F, draw an indefinite line also perpendicular to the directrix, cutting it in D: this will be the direction of the transverse axis; and imagine E and C to be the extremities of that axis. Then, since (Art. 88.) FP : PQ :: FE : ED, and FP : PQ :: FC : CD,

$$\text{FP}+\text{PQ} : \text{FP} :: \text{FD} : \text{FE},$$

and
$$\text{PQ}-\text{FP} : \text{FP} :: \text{FD} : \text{FC}:$$

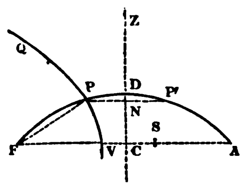
thus, both FE and FC are found; and, consequently, the major axis is obtained.

If FP is less than PQ the curve is an ellipse, if equal a parabola, and if greater an hyperbola.

PROPOSITION XXXIX.

137. To trisect an angle, or a circular arc, by means of an hyperbola.

Let FDA be a given circular arc which is to be trisected: draw the chord FA; and, having bisected it in C, draw an indefinite line CZ perpendicular to FA. Make CV equal to $\frac{1}{3}$FC; then if, with F as a focus, V one extremity of the transverse axis, and CZ as a directrix, there be described (Art. 89. Scholium) an hyperbolic curve VPQ, it will cut the circular arc in P; and the arc FP, will be one third of the given arc FDA.



For (Art. 88.) having joined F, P, and drawn PN perpendicular to CZ,

$$FV : VC :: FP : PN;$$

But, since $VC = \frac{1}{3}FC$, $FV = 2VC$; therefore the straight line $FP = 2PN$.

Now, if PN be produced to cut the arc FDA in P,′ PP′ will be bisected in N, and the arc PDP′, or FDA, in D: thus the arcs FP, PP′, and P′A will be equal to one another, or the arc FDA is trisected in P.

If S represent the centre of the hyperbola, we shall have (Art. 88.)

$$SF : SV :: SV : SC;$$

whence $SF - SV : SV :: SV - SC : SC$, or $FV : SV :: VC : SC$.
But $FV = 2VC$; therefore $SV = 2SC$ and $VC = SC$:
Hence $SV = FV$ and $2SV = SF$, or twice the semi-transverse axis is equal to the excentricity.

SECTION III.

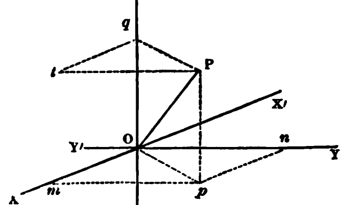
ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

THE EQUATIONS OF LINES AND PLANES IN SPACE.

138. THE position of a point in space is determined when its distances from any three given points are known; but, in order that the varying positions of the point may be expressed algebraically, it is found convenient to imagine the positions of three lines concurring in one point to be given, and to refer the point in space to these lines or to planes conceived to pass through them. The three lines are designated *Axes of the Co-ordinates*, and the planes passing through them are called *Co-ordinate Planes:* these may form any angles, right or oblique, with one another, and may be considered as three faces of a parallelepiped, about one of its solid angles; while the co-ordinate axes may be considered as three concurring edges of the figure. The position of a line and of a plane in space is also indicated by the distances of points in such line or plane from the three co-ordinate planes or axes.

When the situation of one of the co-ordinate planes can be assigned at pleasure, it is convenient to imagine that it coincides with the plane of the paper, the two other planes then making with the paper, and with each other, right or oblique angles. Thus, O being the common intersection of the three axes OX, OY, OZ or the origin of the co-ordinates, OX, OY produced if necessary towards X′ and Y′ may be in the plane of the paper, and the planes ZOX, ZOY inclined to it at any angles, either rising above it, or, if necessary, produced so



as to extend below it. The lines XOX′, YOY′, ZOZ′ are at right angles to one another when the system of co-ordinate planes is rectangular; otherwise the angles which they mutually form are acute or obtuse.

139. Now, in order to express the position of a point P in space, whether the system of axes be rectangular or oblique it is conceived that through the point there pass there planes parallel to the co-ordinate planes, and consequently cutting them in lines parallel to the axes: the distances of these planes, measured on the axes are the co-ordinates of the point. Thus, if the plane P*pm*, P*pn*, P*tq* be parallel to ZOY, ZOX, XOY respectively, the intersections *pn* or O*m* (represented by x), *pm* or O*n* (represented by y), and P*p* or O*q* (represented by z), are the co-ordinates of P.

The point O may be considered as the centre of a sphere whose surface passes through P; and the co-ordinates of P will have positive or negative values depending on the solid sector within which the point is situated. Thus, if P were anywhere within the sector ZXY, the co-ordinates x, y, z would all be considered as positive; if P were within the sector ZYX′, y and z would remain positive, but x would be negative; if P were within the sector Z′XY, x and y would be positive, but z would be negative; and so on. (Art. 4.)

140. The situation of a point P in space may also be expressed by polar co-ordinates, the positions of the co-ordinate axes being given. Thus, imagine a line OP to be drawn to the point from the origin of the co-ordinates; then OP and the three angles XOP, YOP, ZOP being known, it is evident that the position of P will be determined. Or, the system of axes, OX, OY, OZ being for simplicity rectangular, if a plane be conceived to pass through ZO and P cutting the plane XOY in a straight line as O*p*, such plane will be perpendicular to XOY, and then, OP and the angles XO*p*, ZOP being known, the position of P is determined. Since the angles XO*p* and ZOP may each have any values from 0 to 360 degrees, it is evident that the same formula may express the situation of P, in any of the four sectoral spaces between the co-ordinate planes.

141. If a plane passing through a line given in space fall perpendicularly on a plane, the line in which the planes intersect one another, is called the orthogonal projection of the given line. Thus, a straight line in space may be projected orthogonally upon each of the three co-ordinate planes; but generally, when such projection is used, the co-ordinate planes are at right angles to one another.

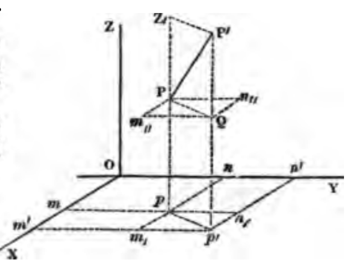
Whether three co-ordinate planes be at right or oblique

angles to one another, lines passing through points in a given line or plane, parallel to one of the co-ordinate planes and intersecting another, form upon this last plane a line or plane which is designated the projection of the given line or plane on the latter. The equations of the lines which on two co-ordinate planes constitute the projections of a line in space, appertain together to this line.

Lines, or planes, passing through a given line and cutting the co-ordinate planes, are called projecting lines, or planes.

PROPOSITION I.

142. To find the distance between any two points in space in terms of the co-ordinates of the points.



Let P and P′ be the given points; and the system of co-ordinate axes being for simplicity rectangular, let fall Pp, P′p' perpendicularly on the plane XOY, and draw pm, $p'm'$ perpendicular to OX; also let the co-ordinates be represented as follow: —

Om by x, mp by y, Pp by z,
Om' by x', $m'p'$ by y', P′p' by z'.

Imagine the lines $m_{,}p\,n$ to be drawn parallel to OX; and $mpn_{,}$ parallel to OY; and let the axes OX, OY, OZ be moved parallel to themselves till O coincides with P. Let the axes OX, OY in their new positions be intercepted by planes passing through P′ parallel to ZOY, ZOX, and imagine P$m_{,,}$, P$n_{,,}$, and P′Q (=PZ$_{,}$) to be new co-ordinates of P′, so that P$m_{,,}$ or Q$n_{,,}$=$pm_{,}$, P$n_{,,}$ or Q$m_{,,}$ =$m_{,}p'$: then

$$\text{P}m_{,,}=x'-x,\ \text{Q}m_{,,}=y'-y,\ \text{and P}'\text{Q}=z'-z;$$

whence (Euc. I. 47.) $\text{P}'\text{P}^2 = (\text{P}m_{,,}^2+\text{Q}m_{,,}^2+\text{P}'\text{Q}^2)$

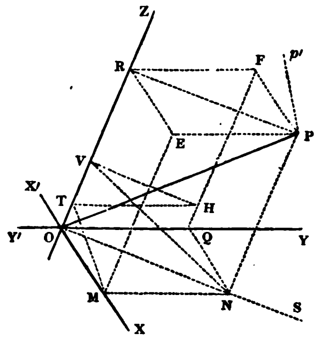
$$=(x'-x)^2+(y'-y)^2+(z'-z)^2,$$

P′ being situated anywhere with respect to P.

If one of the points, as P′, coincides with O the origin of the co-ordinates, we have $x'=0$, $y'=0$, $z'=0$; and the square of the line, or OP^2, becomes $x^2+y^2+z^2$.

Thus the line PP′ is the diagonal of a rectangular parallelepiped, whose sides are P$m_{,,}$, Q$m_{,,}$, and P′Q; and OP is the diagonal of such a parallelepiped, whose sides are Om, mp, or On, and Pp.

a. If the co-ordinate planes ZOX, ZOY, XOY are oblique to one another, imagine planes to pass through P the given point, parallel to those planes cutting ZOX in EM and ER, ZOY in FQ and FR, and XOY in NM and NQ; and let two of the planes intersect one another in PN; also, imagining O, N to be joined, let ON be produced towards S; then $OM = x$, $OQ = y$, $OR = z$.



Now (Pl. Trigon. Art. 57. *b.*)

$$OP^2 = NP^2 + ON^2 + 2NP.ON \cos. ZON,$$

and $$ON^2 = OM^2 + MN^2 + 2OM.MN \cos. XOY.$$

Imagine NV to be let fall perpendicularly on OZ; then

$$ON \cos. ZON = OV.$$

Also, imagine MT to be drawn perpendicular to OZ, TH parallel to OY, and HV perpendicular to OZ; then

$OT = OM \cos. ZOX$, $TV = TH \cos. ZOY$ $(= MN \cos. ZOY)$, and OV, or $ON \cos. ZON, = OM \cos. ZOX + MN \cos. ZOY$; therefore, substituting,

$$OP^2 = NP^2 + OM^2 + MN^2 + 2OM.MN \cos. XOY + 2NP(OM \cos. ZOX + MN \cos. ZOY),$$

or $$OP^2 = z^2 + x^2 + y^2 + 2xy \cos. XOY + 2zx \cos. ZOX + 2zy \cos. ZOY.$$

143. COR. From this proposition it may be proved, that the sum of the squares of the four diagonals of the parallelepiped RN is equal to four times the sum of the squares of the three edges about one of the solid angles; that is,

$$OP^2 + NR^2 + MF^2 + EQ^2 = 4(OM^2 + OQ^2 + OR^2).$$

We have already, as above, the value of OP^2. Now suppose the parallelepiped to be formed between the axes XO produced towards X′, and YO; then the angles YOX′, ZOX′ will be obtuse, while ZOY remains acute; and on merely changing the signs of the terms containing cos. XOY, cos. ZOX in the last value of OP^2, we have

$$NR^2 = z^2 + x^2 + y^2 - 2xy \cos. XOY - 2xz \cos. ZOX + 2zy \cos. ZOY.$$

Again, suppose the parallelepiped to be formed between the axes YO produced towards Y′, and OX; then the angles XOY′, ZOY′ become obtuse, while ZOX remains acute; and

changing the signs of cos. XOY, cos. ZOY in the value of OP^2, we have

$$\mathrm{MP}^2=z^2+x^2+y^2-2xy\cos.\mathrm{YOX}+2xz\cos.\mathrm{ZOX}-2zy\cos.\mathrm{ZOY}.$$

Lastly, suppose the parallelepiped to be formed between the axes YO produced towards Y′, and XO produced towards X′; then the angles ZOY′, ZOX′ are obtuse, while X′OY′ is acute; therefore, changing the signs of cos. ZOY, cos. ZOX in the value of OP^2, we have

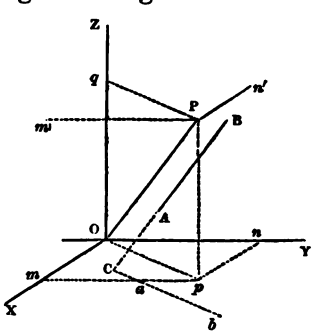
$$\mathrm{EQ}^2=z^2+x^2+y^2+2xy\cos.\mathrm{XOY}-2xz\cos.\mathrm{ZOX}-2zy\cos.\mathrm{ZOY}.$$

Adding together the equivalents of the squares of the four diagonals, all the trigonometrical terms destroy one another, and their sum is $4\,(z^2+x^2+y^2)$, which was to be proved.

If P and P′ be any two points in space, and the co-ordinates of P′ be represented by $x'-x$, $y'-y$, and $z'-z$, the value of $\mathrm{P'P}^2$ will be obtained by merely substituting in the expression for OP^2, $x'-x$ for x, $y'-y$ for y, and $z'-z$ for z.

Proposition II.

144. To find the equations of a line, in space, when the line is projected on three rectangular co-ordinate planes.



Let the line, as OP, pass through the origin of the co-ordinates; and, planes being supposed to pass through P perpendicularly to the co-ordinate planes, let the co-ordinates of P be Om ($=x$), On ($=y$), Oq ($=z$); also, let the angles XOP, YOP, ZOP, which OP makes with the axes, be represented by α, β, γ. Again, imagine planes to pass through OP perpendicularly to the co-ordinate planes, so as to cut ZOX in Om' ZOY in On', and XOY in Op, in which case Om', On', Op are the projections of OP on the co-ordinate planes; then we shall have

$$\tan.\mathrm{ZO}m'=\frac{\mathrm{O}m}{\mathrm{O}q}=\frac{x}{z}=\frac{\cos.\alpha}{\cos.\gamma},$$

$$\tan.\mathrm{ZO}n'=\frac{\mathrm{O}n}{\mathrm{O}q}=\frac{y}{z}=\frac{\cos.\beta}{\cos.\gamma},$$

and $$\tan. \text{XO}p = \frac{mp}{\text{O}m} = \frac{y}{x} = \frac{\cos. \beta}{\cos. \alpha}.$$

Thus the equations of OP, that is, of its projections on ZOX, ZOY, become (Art. 9.)

$$\frac{\cos. \alpha}{\cos. \gamma} z = x, \quad \frac{\cos. \beta}{\cos. \gamma} z = y,$$

which may be put in the forms

$$az = x, \quad bz = y.$$

If the original line, as AB, do not pass through the origin of the co-ordinates, imagine a line, as OP, parallel to it, to be drawn through the point O; then, the lines being parallel, two planes which are parallel to one another may be conceived to pass through them, and these parallel planes, being made to intersect either of the co-ordinate planes, the lines of section, as ab and $\text{O}p$ on the plane XOY, will (Geom. Planes, Prop. 16.) be parallel to one another. In the present proposition, the planes passing through the two lines are supposed to be perpendicular to the plane of projection, and the lines of section are the projections of those lines. It follows that the equations of a line in space, not passing through the origin of the co-ordinates, may (Art. 10.) be represented on the planes ZOX, ZOY respectively by

$$x = az + h, \text{ and } y = bz + k. \qquad (a)$$

From these equations may be obtained the equation of the line projected on XOY; for, eliminating z, we have

$$y = \frac{b}{a} x + \frac{ak - bh}{a}, \qquad (b)$$

which may be represented (m denoting $\frac{b}{a}$ or tan. XOp) by

$$y = mx + n.$$

a. If the line do not pass through the origin of the co-ordinates, the point, as C, in which it intersects the plane XOY, is found by making $z = 0$ in the equations (a), which gives, for the position of the point of intersection, $x = h$, $y = k$. Making $x = 0$ in the equations (a) and (b), we have the values of z and y, which determine the point of intersection on ZOY; and making $y = 0$ in the same equations, we have the values of z and x for the point of intersection on ZOX.

b. The equations for a line in space, when projected on co-ordinate planes oblique to one another, may, if the lines

pass through the origin of the co-ordinates, be represented on ZOX, ZOY (Fig. to Art. 142. *a.*) by

$$x=az, \quad y=bz;$$

and if it do not pass through the origin of the co-ordinates, by the equations

$$x=az+h, \quad y=bz+k:$$

in all these equations (Art. 9.)

$$a=\frac{\sin. \text{OEM}}{\sin. \text{EOM}}, \text{ and } b=\frac{\sin. \text{OFQ}}{\sin. \text{FOQ}}.$$

PROPOSITION III.

145. To find the polar equations of a line in space, the co-ordinate axes being at right angles to one another.

Let the line, as OP (Fig. to Art. 144.), pass through the origin of rectangular co-ordinates, and let it be represented by r; then, the angles which OP makes with the co-ordinate axes being represented as before, we have for the polar equations,

$$x=r \cos. \alpha, \quad y=r \cos. \beta, \quad z=r \cos. \gamma.$$

But the equations of the projected lines Om', On' being represented by $x=az$, $y=bz$ (in which a and b are supposed to denote, respectively, tan. ZOm' and tan. ZOn'), on substituting these values of x and y in the equation

$$r^2 \; (=\text{OP}^2)=x^2+y^2+z^2,$$

the latter becomes $r^2=(a^2+b^2+1)\,z^2$;

whence $$z=\pm\frac{r}{\sqrt{(a^2+b^2+1)}}.$$

This value of z being substituted in the same equations for x and y, we obtain

$$x=\pm\frac{a\,r}{\sqrt{(a^2+b^2+1)}},$$

and $$y=\pm\frac{b\,r}{\sqrt{(a^2+b^2+1)}}.$$

Again, if θ be the angle which Op, the projection of OP, makes with OX, we shall have, since $r \sin. \gamma=$Op,

$$x=r \sin. \gamma \cos. \theta,$$

and $$y=r \sin. \gamma \sin. \theta$$

while $$z=r \cos. \gamma.$$

a. If the co-ordinate planes form oblique angles with one another, the polar equations for the line OP may be found thus: —

Let OP (Fig. to Art. 142. *a.*) be represented by r as before, and let the angles which it makes with the planes ZOY, ZOX, and XOY be represented by α', β', and γ'; also let the inclinations of the lines OX, OY, OZ to the planes ZOY, ZOX, XOY be denoted by θ, θ', and θ''.

Imagine $\text{P}p'$ to be let fall perpendicularly on the plane ZOY; then (Trigon. Art. 30. *l.*)

$$r \sin. \alpha' = \text{P}p';$$

also, if a line be let fall from M perpendicularly on ZOY, such line will be equal to $\text{P}p'$; hence

$$x \sin. \theta = \text{P}p'.$$

Therefore $$x \sin. \theta = r \sin. \alpha':$$

whence $$x = \frac{\sin. \alpha'}{\sin. \theta} r.$$

In like manner it may be shown that

$$y = \frac{\sin. \beta}{\sin. \theta'} r \text{ and } z = \frac{\sin. \gamma'}{\sin. \theta''} r.$$

PROPOSITION IV.

146. To find the equations for a line which is to pass through two given points in space, the co-ordinate axes making any angle with one another.

Let the co-ordinates of the two points be x', y', z', and x'', y'', z''; it is evident that, with respect to these points, the equations (a) Art. 144. for the projected lines will be

$$x' = az' + h, \quad y' = bz' + k,$$

and $$x'' = az'' + h, \ y'' = bz'' + k;$$

whence $$x'' - x' = a\,(z'' - z'), \text{ or } a = \frac{x'' - x'}{z'' - z'},$$

and $$y'' - y' = b\,(z'' - z'), \text{ or } b = \frac{y'' - y'}{z'' - z'}.$$

But subtracting the equations for x' and y' from the equations (a) Art. 144. we have

$$x - x' = a\,(z - z'), \text{ and } y - y' = b\,(z - z'):$$

substituting in these the above values of a and b, the required equations are

$$x-x'=\frac{x''-x'}{z''-z'}(z-z'),$$

and

$$y-y'=\frac{y''-y'}{z''-z'}(z-z');$$

in which x, y, z represent the co-ordinates of any point whatever in the line.

Proposition V.

147. To find the co-ordinates of a point in which two lines whose equations are given intersect one another.

Let the given equations be

$$x=az+h,\ y=bz+k$$
$$x'=a'z'+h',\ y'=b'z'+k'.$$

At the point of intersection $x=x'$, $y=y'$ and $z=z'$;

therefore $(a-a')z+h-h'=0$

and $(b-b')z+k-k'=0$;

whence $z=\frac{h'-h}{a-a'}$, or $=\frac{k'-k}{b-b'}$;

substituting the first of these values of z in the equation for x, we have

$$x=a\frac{h'-h}{a-a'}+h=\frac{ah'-a'h}{a-a'};$$

and the other in the equation for y, we have

$$y=b\frac{k'-k}{b-b'}+k=\frac{bk'-b'k}{b-b'}.$$

These values of x, y, and z are the co-ordinates of the point of intersection. If $a=a'$ and $b=b'$ the values of x, y, and z are infinite; which proves that the lines are then parallel to one another.

Proposition VI.

148. To find the equations for a straight line passing through a given point, and making given angles with three rectangular co-ordinate axes.

Imagine a line, as OP (Fig. to Art. 144.) passing, through the origin, and making with the co-ordinate axes angles equal to those which the given line makes with the same axes, this line will be parallel to the given line.

Let the angles made with OX, OY, OZ be α, β, γ; then x', y', z' being the co-ordinates of the given point, the equations of the required line, on ZOX, ZOY, will be as in the equations (a) Art. 144., and substituting for a and b their trigonometrical equivalents,

$$\frac{\cos.\ \alpha}{\cos.\ \gamma}z'+h=x', \quad \text{and } \frac{\cos.\ \beta}{\cos.\ \gamma}z'+k=y';$$

whence $$h=x'-z'\frac{\cos.\ \alpha}{\cos.\ \gamma}, \text{ and } k=y'-z'\frac{\cos.\ \beta}{\cos.\ \gamma}.$$

Therefore x, y, z being the co-ordinates of any other point in the line, the required equations for the projections on ZOX, ZOY, by the substitution of these values of h and k in the equations

$$\frac{\cos.\ \alpha}{\cos.\ \gamma}z+h=x, \text{ and } \frac{\cos.\ \beta}{\cos.\ \gamma}z+k=y,$$

will be $$\frac{\cos.\ \alpha}{\cos.\ \gamma}(z-z')=x-x' \text{ and } \frac{\cos.\ \beta}{\cos.\ \gamma}(z-z')=y-y'.$$

149. COR. If it were required to find the equations for a straight line passing through a given point and parallel to a given line:

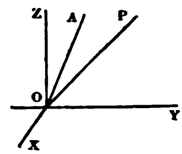
Through the origin of the co-ordinates a line may be imagined to pass parallel to the given line, consequently making equal angles with the co-ordinate axes; then, the co-ordinates of the given point being x', y', z', the required equations for the line will be, on ZOX and ZOY, the same as those in the proposition.

PROPOSITION VII.

150. To find the angle contained between two lines which intersect one another in space, in terms of the angles which they make with the co-ordinate axes.

If the intersection should not take place at the origin of the co-ordinates, conceive two lines to be drawn through that point parallel to the given lines; the angle contained between these will be equal to that which is contained between the original lines.

Let OA, OP be the two lines drawn from O, and let their lengths be represented by r and r'. Let the rectilinear co-ordinates of A be x, y, z, and those of P be x', y', z'; again, let the inclinations of the lines OA and OP to the axes OX, OY, OZ be represented by α, β, γ, and α', β', γ'; and let the required angle be represented by θ.



Then (Pl. Trig. Art. 57. *b.*), A, P being joined.

$$AP^2 = r^2 + r'^2 - 2\, r r' \cos. \theta;$$

also (Art. 142.) $AP^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2$; expanding the second member of the last equation, we have

$$AP^2 = x'^2 + y'^2 + z'^2 + x^2 + y^2 + z^2 - 2\,(x'x + y'y + z'z),$$
$$= r'^2 + r^2 - 2\,(x'x + y'y + z'z);$$

therefore $\qquad r r' \cos. \theta = x'x + y'y + z'z. \qquad (c)$

But $\qquad x = r \cos. \alpha, \quad y = r \cos. \beta, \quad z = r \cos. \gamma,$

and $\qquad x' = r' \cos. \alpha', \quad y' = r' \cos. \beta', \quad z' = r' \cos. \gamma';$

therefore, substituting in the last equation,

$$\cos. \theta = \cos. \alpha \cos. \alpha' + \cos. \beta \cos. \beta' + \cos. \gamma \cos. \gamma'.$$

When the two lines are parallel to one another, $\theta = 0$ and $\cos. \theta = 1$; therefore the equation becomes

$$1 = \cos. \alpha \cos. \alpha' + \cos. \beta \cos. \beta' + \cos. \gamma \cos. \gamma'.$$

When the two lines are at right angles to one another, $\theta = 90°$ and $\cos. \theta = 0$; therefore

$$0 = \cos. \alpha \cos. \alpha' + \cos. \beta \cos. \beta' + \cos. \gamma \cos. \gamma'.$$

151. COR. If it were required to express the angle contained between two lines, as OA, OP, in terms of the angles which their projections on two of the three co-ordinate planes make with the line in which those planes intersect one another, let x, y, z, and x', y', z' be co-ordinates of A and P, or of any two points, one in each line. Then the equations for the two lines being,

on ZOX, $\qquad x = az, \quad x' = a'z';$

on ZOY, $\qquad y = bz, \quad y' = b'z';$

in which a, a', b, b' (the systems of co-ordinate axes being rectangular) are the tangents of the angles which the projections of the given lines on ZOX, ZOY make with OZ.

Substituting these values of x, x', y, y' in the value of cos. θ, at (c) above. viz.

$$\frac{xx'+yy'+zz'}{rr'}, \text{ or } \frac{xx'+yy'+zz'}{\sqrt{\{(x^2+y^2+z^2)(x'^2+y'^2+z'^2)\}}};$$

the latter becomes, on dividing all the terms by zz',

$$\frac{aa'+bb'+1}{\sqrt{\{(a^2+b^2+1)(a'^2+b'^2+1)\}}}. \qquad (c')$$

PROPOSITION VIII.

152. To find the equation for a plane in space, in terms of rectangular co-ordinates.

Imagine a line to be drawn (Fig. to Art. 150.) from O, the origin of the co-ordinates, perpendicularly on the plane, meeting the latter in A, and let P be any point in the plane. Join O, P and A, P, (AP will be at right angles to OA); and let the co-ordinates of A be a, b, c, while those of P are x, y, z.

Then, as in Art. 142., $\quad OA^2=a^2+b^2+c^2,$

$$OP^2=x^2+y^2+z^2;$$

and $\quad AP^2=(x-a)^2+(y-b)^2+(z-c)^2;$

therefore $\quad OA^2\ (=OP^2-AP^2)=2\,(ax+by+cz)-(a^2+b^2+c^2),$

or $\quad OA^2=2\,(ax+by+cz)-OA^2;$

or again, representing OA by D,

$$ax+by+cz=D^2, \qquad (d)$$

which is the equation for the plane.

But, if the angles which OA makes with OX, OY, OZ be represented by α, β, γ, we shall have

$$a=D\cos.\alpha,\quad b=D\cos.\beta,\quad c=D\cos.\gamma;$$

and the equation (d) becomes

$$x\cos.\alpha+y\cos.\beta+z\cos.\gamma=D, \qquad (d')$$

which is usually put in the form

$$Ax+By+Cz=D. \qquad (d'')$$

It is evident that, when the plane passes through the origin of the co-ordinates, $D=0$; and the equations (d) and (d') become, respectively,

$$ax+by+cz=0,$$

and $\quad x\cos.\alpha+y\cos.\beta+z\cos.\gamma=0.$

The inclination of two planes to one another is equal to the angle contained between two lines which are respectively perpendicular to the planes; therefore, if a line, as O A, be drawn from O, the origin of three rectangular co-ordinates, perpendicularly to any plane in space, since OX, OY, OZ are respectively perpendicular to the co-ordinate planes ZOY, ZOX, XOY, the angles XOA ($=\alpha$), YOA ($=\beta$), ZOA ($=\gamma$) which the line O A makes with the axes, will also denote the angles which the plane, in space, makes with those co-ordinate planes. Thus the equation (d') is the equation for a plane, in space, in terms of the inclinations of the plane to the three co-ordinate planes.

153. COR. 1. If the line O P were conceived to coincide with O A, which may represent any line passing through the origin of the co-ordinates, the equation (d') would become

$$a \cos. \alpha + b \cos. \beta + c \cos. \gamma = \text{D}.$$

Now, since the terms in the first member of this last equation are respectively the projections of a, b, c, on the line O A, by lines drawn perpendicularly on the latter from their extremities; it follows that, if any line in space be projected on three rectangular co-ordinate axes, and the projected lines be again projected on the original line, the sum of these second projections will be equal to that line.

154. COR. 2. If the plane in space cut the co-ordinate plane X O Y; on the line of intersection $z=0$, and the equation (d'') becomes

$$\text{A}x + \text{B}y = \text{D}.$$

Again, if the plane cut ZOX, or ZOY, we have, since on the line of intersection $y=0$, or $x=0$,

$$\text{A}x + \text{C}z = \text{D}, \text{ or } \text{B}y + \text{C}z = \text{D};$$

and these are the three several equations for lines of intersection with the three co-ordinate planes.

From the first of these equations we have

$$y = \frac{\text{D}}{\text{B}} - \frac{\text{A}}{\text{B}} x,$$

in which $\frac{\text{D}}{\text{B}}$ is the distance from O to the point in which O Y is cut by the intersection of the plane in space with X O Y, and $\frac{\text{A}}{\text{B}}$ or $\frac{\cos. \alpha}{\cos. \beta}$ is the cotangent of the angle which O X makes with the projection of O A on X O Y, or the tangent of the angle which the intersection of the plane with X O Y

makes with OX. The like significations have the terms which enter into the values of x and z, which may be deduced from the two other equations for D.

a. It follows that, if two planes in space are parallel to one another, the two lines in which they intersect each of the co-ordinate planes are parallel to one another. For α, β, γ (Art. 150.) being the same for both planes, the values of $\frac{\cos. \alpha}{\cos. \beta}$, &c., or the tangents of the angles which the intersections of both planes make with OX, &c., are respectively equal to one another; therefore the angles are equal, and the lines are parallel to one another. The parallelism of the lines is, however, manifest from Elementary Geometry (Geom. Planes, Prop. XIV.).

b. It follows also that, when two straight lines in space are parallel to one another, their projections on each of the co-ordinate planes are parallel to one another. For the two lines being parallel, they may be conceived to lie in two planes which are parallel to one another, and have any position with respect to the co-ordinate planes: let each of these planes be perpendicular to either of the co-ordinate planes; then their intersections with the latter plane will be the projected lines, on that plane; and the projecting planes being parallel to one another, the two projected lines on each co-ordinate plane will, as above, be parallel to one another.

c. It is evident from what has been said, that if, on any plane which intersects either of the co-ordinate planes, a line be drawn parallel to the intersection of the planes; the projection of the line on that co-ordinate plane will be parallel to the line of intersection.

155. COR. 3. If a plane in space cut the axis OX, both $y=0$ and $z=0$; and we have, from (d''), $\text{A}x-\text{D}=0$: a like equation is obtained if the plane cut OY or OZ; in the former case, $x=0$, $z=0$; and in the latter, $x=0$, $y=0$.

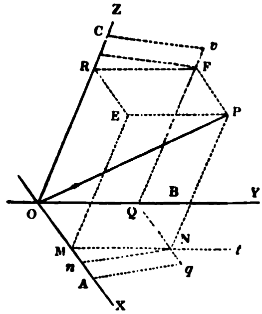
156. COR. 4. If a plane be parallel to XOY, the projection of the plane on ZOX, or ZOY, is a straight line parallel to OX, or OY; and, for these projections, since either $y=0$, or $x=0$, and z is independent of both, we get, from (d''), $\text{C}z-\text{D}=0$. A corresponding equation will be obtained for the projections of a plane parallel to either of the other co-ordinate planes.

PROPOSITION IX.

157. To find the equation for a plane in space, with respect to three co-ordinate axes forming any angles with one another.

The equation for OP^2 (Art. 142. *a.*) may be considered as the equation for a plane in space, in which plane P is any point; thus the co-ordinates of P being x, y, z, as in that Article: —

From P let fall PA perpendicularly on OX, and Pq perpendicularly on QN produced; these lines being perpendicular to PF, which is parallel to OX and QN, will be in a plane perpendicular to PF, and consequently to the plane PR, or to XOY, which is parallel to PR. Join A, q, and draw Nn parallel to Aq; then both Aq and Nn will be perpendicular to OX.



$$\text{Now } Mn\ (= MN \cos. NMn) = y \cos. XOY,$$

$$Nq \text{ or } An\ (= PN \cos. PNq) = z. \cos. ZOX,$$

and $OA\ (= OP \cos. XOP)$;

therefore

$$OA\ (= OP \cos. XOP) = x + y \cos. XOY + z \cos. ZOX.$$

In like manner, letting fall PB perpendicularly on OY, and Pt on MN produced, we should have

$$OP \cos. YOP = y + x \cos. XOY + z \cos. ZOY.$$

Also, letting fall PC perpendicularly on OZ, and Pv on QF produced, we should obtain

$$OP \cos. ZOP = z + y \cos. ZOY + x \cos. ZOX.$$

Multiplying the first of the three last equations by x, the second by y, and the third by z, it will be found that the sum of the second members, when so multiplied, is identical with the second member of the equation for OP^2 (Art. 142. *a.*); therefore

$$OP^2 = x \,.\, OP \cos. XOP + y \,.\, OP \cos. YOP + z \,.\, OP \cos. ZOP,$$

or, $$OP = x \cos. XOP + y \cos. YOP + z \cos. ZOP.$$

This is the equation for a plane, whether the co-ordinate axes be rectangular or oblique, P being any point in the plane, and x, y, z the co-ordinates of that point.

From the last equation it is evident that, if the projections of any line, as OP, on the oblique co-ordinate axes, be orthogonally projected on the same line, the sum of the last pro-

jections will be equal to that line. This circumstance corresponds to the inference in Art. 153. respecting the projections of a line on rectangular co-ordinate axes.

PROPOSITION X.

158. To find the point in which a given straight line will intersect a given plane.

Let the equations of the line, supposed to be projected on the rectangular co-ordinate planes ZOX, ZOY, be

$$x = az + h, \quad y = bz + k.$$

Now, at the point of intersection, the co-ordinates x, y, and z are the same both for the line and the plane; therefore, substituting these values of x and y in the equation of the plane [(d'') Art. 152.],

$$\text{A}(az+h) + \text{B}(bz+k) + \text{C}z = \text{D}:$$

whence $$z = -\frac{\text{A}h + \text{B}k - \text{D}}{\text{A}a + \text{B}b + \text{C}};$$

and substituting this value of z in the equations for x and y, there will be obtained the equivalents of those co-ordinates. Thus there will be found the values of x, y, and z for the required point of intersection.

COR. If the line were parallel to the given plane, the point of intersection might be considered as infinitely remote, or x, y, z would be infinite; therefore the denominators in the values of these co-ordinates would be equivalent to zero. Hence, from the above value of z we should have

$$\text{A}a + \text{B}b + \text{C} = 0.$$

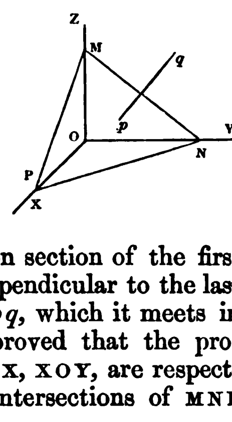
CHAP. II.

INCLINATIONS OF LINES AND PLANES TO ONE ANOTHER.

PROPOSITION I. — LEMMA.

159. IF a line, in space, be perpendicular to any plane, the orthogonal projection of the line on either of three rectangular co-ordinate planes will be perpendicular to the intersection of the plane with that co-ordinate plane.

Let MP, MN be the lines in which the plane intersects the co-ordinate planes ZOX, ZOY, and let pq be the projection of the perpendicular line upon ZOY. Then, since the projecting plane passing through the original line is perpendicular to ZOY, and by passing through that original line it is perpendicular to the plane MNP; it follows that the two planes ZOY and MNP are perpendicular to the projecting plane: hence MN, the common section of the first planes is (Geom. Planes, Prop. XIX.) perpendicular to the last plane, and therefore perpendicular to pq, which it meets in that plane. In like manner it may be proved that the projections of the perpendicular line on ZOX, XOY, are respectively perpendicular to MP and PN, the intersections of MNP with those planes.



PROPOSITION II.

160. To find the equations for a line passing through a given point and perpendicular to a given plane.

Let the co-ordinates of the given point be x', y', z', and those of the point in which the line intersects the plane be x, y, z. Now, the equation for the plane being (Art. 152.)

$$Ax + By + Cz = D;$$

if lines on the plane, parallel to the intersections of the latter

with the co-ordinate planes ZOX, ZOY, and passing through the point whose co-ordinates are x, y, z, be projected on those planes; the lines so projected, being (Art. 154. c.) parallel to the intersections, their equations will be,

$$\text{on ZOX,}\quad Ax + Cz = D, \text{ whence } x = -\frac{C}{A}z + \frac{D}{A};$$
$$\text{and on ZOY,}\quad By + Cz = D, \text{ whence } y = -\frac{C}{B}z + \frac{D}{B}. \qquad (e)$$

But the projections of the perpendicular line on ZOX, ZOY are, by the preceding lemma, perpendicular to the intersections of the given plane with the co-ordinate planes, and to the lines represented by the equations (e); therefore the equations for the projections of the perpendicular line (x, y, z being the co-ordinates of its intersection with the given plane) will be (Art. 16. a.)

$$x = \frac{A}{C}z + P, \text{ and } y = \frac{B}{C}z + Q \qquad (f)$$

(P and Q being put for the constant terms). And the equations for the projections of the same line, at points which are the projections of that point whose co-ordinates are x', y', z', will be

$$x' = \frac{A}{C}z' + P, \qquad y' = \frac{B}{C}z' + Q.$$

Therefore, by subtraction,

$$x' - x = \frac{A}{C}(z' - z),$$
$$\text{and}\qquad y' - y = \frac{B}{C}(z' - z). \qquad (g)$$

These are the equations for the perpendicular to the given plane when that perpendicular is projected on the planes ZOX, ZOY.

The values of x and y determined from the equations (g) being substituted in the equation of the plane, from the latter the value of z may be determined; and thus the point is found at which a line perpendicular to the plane, and passing through a given point, will intersect the plane.*

* Since in the equation $Ax + By + Cz = D$, for a plane (AO, or D, Fig. to Art. 150., being a line let fall from O perpendicularly on the plane)

$$A = \cos. AOX, \quad B = \cos. AOY, \quad C = \cos. AOZ;$$

161. COR. The length of the perpendicular between the points whose co-ordinates are x, y, z and x' y' z' is evidently equal to

$$\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}$$

Now, if r represent the length of a perpendicular let fall from O on the plane, and p be the length of the perpendicular last mentioned, we shall have [(d) Art. 152.]

$$ax+by+cz=r^2,$$

and
$$ax'+by'+cz'=(r+p)^2;$$

whence
$$a(x'-x)+b(y'-y)+c(z'-z)=2rp+p^2.$$

Substituting in this equation the values of $x'-x$ and $y'-y$ in (g) we shall have,

$\frac{A}{C}$ and $\frac{B}{C}$ being respectively identical with $\frac{a}{c}$ and $\frac{b}{c}$,

$$z'-z=\frac{c(2pr+p^2)}{a^2+b^2+c^2};$$

from which, by substitution in (g), we shall obtain corresponding values of $x'-x$ and $y'-y$. Lastly, substituting all these values in the radical expression above, we obtain

$$p=\pm\frac{ax'+by'+cz'-r^2}{\sqrt{(a^2+b^2+c^2)}}.$$

PROPOSITION III.

162. To find the inclination of a given line to a plane.

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it follows that, in the equations (f) and (g),

$$\frac{A}{C}=\frac{\cos. AOX}{\cos. AOZ}\left(=\frac{x}{z}\right), \quad \text{and } \frac{B}{C}=\frac{\cos. AOY}{\cos. AOZ}\left(=\frac{y}{z}\right);$$

in which $\frac{x}{z}$ is manifestly the tangent of the angle which the projection on ZOX of a perpendicular to the plane makes with OZ, and $\frac{y}{z}$ is the tangent of the angle which the projection on ZOY of a perpendicular to the plane makes with the same line OZ; or $\frac{x}{z}$, $\frac{y}{z}$ are the cotangents of the angles which the projections of the perpendicular to the plane make with OX and OY respectively. The coefficients of z in the equations (e) are evidently tangents of the angles which the intersections of the given plane with the co-ordinate planes make with OZ; or they are respectively the cotangents of the angles which those intersections make with XO and YO produced.

Imagine a line, as O A (Fig. to Art. 150.), to be drawn from the origin of the rectangular co-ordinates perpendicularly on the given plane, and let O P be parallel to the given line; then the angle A O P will be equal to the complement of the required inclination of the line to the plane. Now, if the equation for the given plane be represented by

$$\text{A}x+\text{B}y+\text{C}z=\text{D}, \qquad [(d'')\ \text{Art. } 152.]$$

the equations for a line, as O A, perpendicularly to it, if the perpendicular be supposed to pass through the origin of the co-ordinates, will be [(f) Art. 158.)]

on Z O X, $\qquad x=\frac{\text{A}}{\text{C}}z$

and on Z O Y, $\qquad y=\frac{\text{B}}{\text{C}}z$:

also the equations of the given line, supposed to pass through O, will be, on Z O X, $x'=a'z'$, and on Z O Y, $y'=b'z'$.

In these equations $\frac{\text{A}}{\text{C}}$ and $\frac{\text{B}}{\text{C}}$ are the quantities designated, respectively, a and b in the equations for x and y (Art. 151.), while a' and b' have the same signification as have a' and b' in the values of x' and y' in that Article. Therefore, on substituting $\frac{\text{A}}{\text{C}}$ for a and $\frac{\text{B}}{\text{C}}$ for b in the formula (c') in the same Article, that formula will become an expression for the cosine of the angle between the given line and a perpendicular to the plane; and consequently for the sine of the required angle between the given line and the plane.

If the line were parallel to the plane; since the sine of the angle between the line and the plane would then be zero, the numerator of the expression (c') would be zero, or we should have

$$aa'+bb'+1=0;$$

whence, by the substitution, just mentioned,

$$\text{A}a'+\text{B}b'+\text{C}=0.$$

Proposition IV.

163. To find the inclination of a given plane to each of three rectangular co-ordinate planes.

Since the angle which two planes make with one another is equal to the angle contained between two lines which are perpendicular to the planes, and intersect one another; it

follows that, if a line pass through the origin of the co-ordinates perpendicularly to the given plane, the angles which it makes with OX, OY, OZ will, respectively, be equal to the angles which the given plane makes with the planes ZOY, ZOX, XOY.

Now the equation for the plane being [(*d*) Art. 152.] represented by

$$ax+by+cz=r^2 \ (=a^2+b^2+c^2)$$

in which r is the length of a perpendicular let fall on the plane from the origin of the co-ordinates; it follows that $\pm\frac{a}{r}$ denotes the cosine of the angle which the perpendicular makes with OX; it is therefore equal to the cosine of the angle which the given plane makes with ZOY. In like manner, the cosine of the angle which the plane makes with ZOX is denoted by $\pm\frac{b}{r}$, and the cosine of that which it makes with XOY by $\pm\frac{c}{r}$.

PROPOSITION V.

164. **The inclinations of two planes to each of three rectangular co-ordinate planes being given; to find the inclination of the planes to one another.**

Imagine a line to be drawn through the origin of the co-ordinates perpendicular to each of the given planes; then the angle contained between these perpendiculars will be equal to the required angle of inclination. Now let x, y, z be the co-ordinates of any point in one of the perpendiculars, and x', y', z' the co-ordinates of any point in the other; then θ representing the required inclination, we have [(*c*) Art. 150.]

$$\cos.\theta=\frac{xx'+yy'+zz'}{rr'}, \text{ or } =\frac{xx'+yy'+zz'}{\surd\{(x^2+y^2+z^2)(x'^2+y'^2+z'^2)\}}$$

which, in terms of the tangents of the angles made by the projections of the perpendiculars on the co-ordinate planes, is [(*c'*) Art. 151.]

$$\cos.\theta=\frac{aa'+bb'+1}{\surd\{(a^2+b^2+1)(a'^2+b'^2+1\}}.$$

When one of the planes is at right angles to the other, $\cos.\theta=0$;

whence $\qquad xx'+yy'+zz'=0,$

or $\qquad aa'+bb'+1 \ =0.$

PROPOSITION VI.

165. To find the equation for a plane which shall pass through a given point, and be parallel to a given plane.

Let x, y, z be the co-ordinates of any point in the required plane, and let α, β, γ be the angles which a perpendicular to both the planes makes with OX, OY, OZ; then [(d') Art. 152.]

$$x \cos.\ \alpha + y \cos.\ \beta + z \cos.\ \gamma = r,$$

(r being the length of the perpendicular let fall on the required plane from the origin of the co-ordinates).

Now, since the required plane is to pass through a given point, if the co-ordinates of that point be x', y', z', we shall have also

$$x' \cos.\ \alpha + y' \cos.\ \beta + z' \cos.\ \gamma = r;$$

therefore, by subtraction,

$$(x'-x) \cos.\ \alpha + (y'-y) \cos.\ \beta + (z'-z) \cos.\ \gamma = 0,$$

which is the equation for the required plane.

Note.—The proposition might have been enunciated thus: — To determine a plane passing through a given point (x', y', z') perpendicularly to a given line. Or, To determine a plane passing through a given point (x', y', z') making given angles (α, β, γ) with the co-ordinate planes.

PROPOSITION VII.

166. To determine the line in which two given planes intersect one another.

Let α, β, γ be the inclinations of one of the planes, and α', β', γ' be the inclinations of the other plane to the three rectangular co-ordinate planes: also, let x', y', z' be given co-ordinates of a point in one of the planes, and x'', y'', z'' be given co-ordinates of a point in the other; and let x, y, z be co-ordinates of a point in the line of section. Then, for one of the planes we have (Art. 165.)

$$(x'-x) \cos.\ \alpha + (y'-y) \cos.\ \beta + (z'-z) \cos.\ \gamma = 0,$$

and for the other,

$$(x''-x) \cos.\ \alpha' + (y''-y) \cos.\ \beta' + (z''-z) \cos.\ \gamma' = 0;$$

or, dividing by cos. γ and cos. γ' respectively,

$$(x'-x)\frac{\cos.\ \alpha}{\cos.\ \gamma}+(y'-y)\frac{\cos.\ \beta}{\cos.\ \gamma}+z'-z=0,$$

and $$(x''-x)\frac{\cos.\ \alpha'}{\cos.\ \gamma'}+(y''-y)\frac{\cos.\ \beta'}{\cos.\ \gamma'}+z''-z=0.$$

The trigonometrical coefficients being represented in the first equation by a and b, and in the second by a' and b' (these letters representing the tangents of the angles which the projections of perpendiculars to the planes, on ZOX and ZOY, make with OZ, Art. 144.), the equations may be put in the form

$$ax+by+z=\text{M},$$

and $$a'x+b'y+z=\text{N};$$

M and N being put for the sums of all the given terms when transposed to the second members. Then, multiplying the first equation by a', and the second by a, in order to eliminate x, we have, after subtraction,

$$(a'b-ab')\,y+(a'-a)\,z=a'\text{M}-a\text{N}.$$

Again, multiplying the first equation by b', and the second by b, in order to eliminate y, we get

$$(ab'-a'b)\,x+(b'-b)\,z=b'\text{M}-b\text{N}.$$

These are the equations for the line of intersection, that line being projected on ZOY and ZOX.

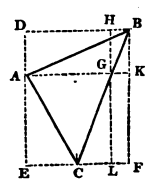
CHAP. III.

PROJECTIONS OF FIGURES ON CO-ORDINATE PLANES.

PROPOSITION I.

167. If any plane figure be orthogonally projected upon a plane surface which is inclined to that of the figure in any angle, the projected figure will be equivalent to the product of the original figure multiplied by the cosine of the inclination.

Any plane figure can be divided into triangles; therefore, ABC being considered as one of the triangles into which the figure is divided, imagine a rectangular parallelogram as EB to be described about it, so that two opposite sides, DE, BF, may be coincident with, or parallel to the line in which the planes intersect one another. Then, if this rectangle be projected on the plane which is inclined to it, the projecting lines being parallel to one another, and perpendicular to the plane on which the figure is to be projected, the projected figure will be a rectangular parallelogram, of which two opposite sides are equal and parallel to DE or BF; and the inclination of the planes being represented by θ, each of the other two opposite sides will be equal to DB cos. θ, or EF cos. θ. It follows that the surface of the rectangle DF, when projected, will be expressed by



$$\text{DE}.\text{DB}\cos.\theta, \text{ or DF}\cos.\theta.$$

Now, if lines be drawn, as in the figure, parallel and perpendicular to DE, there will be formed the rectangular parallelograms EG, DG, &c., each of which has two of its opposite sides coincident with, or parallel to the intersection of the planes. Then, the projection of EG will be expressed by

$$\text{AE}.\text{AG}\cos.\theta, \text{ or EG}\cos.\theta;$$

but (Euc., 41. I.) the triangle ACG$=\frac{1}{2}$EG, and the projection of ACG will be equal to half the projection of EG; therefore the projection of ACG$=$ACG cos. θ.

Also, half the projection of DG$=\frac{1}{2}$DG cos. θ:

but (Euc. 41. I.) the triangle ABG (=AHG)=$\frac{1}{2}$DG, and the projection of ABG or AHG will be equal to half the projection of DG; therefore the projection of ABG=ABG cos. θ.

Hence, by addition, the projection of the triangle ABC= ABC cos. θ.

In like manner may the proof be extended to all the triangles composing a given figure; therefore the area of a projected plane figure is equivalent to the given area multiplied by the cosine of the inclination of the planes.

PROPOSITION II.

168. The square of the area of any plane figure is equivalent to the sum of the squares of its projections on three rectangular co-ordinate planes. [*The square of an area is to be understood as a numerical expression only.*]

Let the inclinations of the plane of the given figure to the co-ordinate planes be α, β, γ; and let the area of the given figure be represented by A; then

$$A \cos. \alpha,\ A \cos. \beta,\ A \cos. \gamma,$$

will be the equivalents of the three projected areas; and the sum of their squares will be

$$A^2 (\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma).$$

But cos. α, cos. β, cos. γ, representing three edges of a rectangular parallelepiped of which the diagonal is unity, we have $\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1$; therefore the sum of the squares of the three projected areas is equivalent to A^2, or to the square of the original area.

COR. It is evident that, if a triangular pyramid have three of its edges at right angles to one another, whatever be the lengths of those edges, the square of the plane side subtending the solid angle formed by those edges will be equivalent to the sum of the squares of the three other plane sides.

PROPOSITION III.

169. Let three rectangular co-ordinate planes represented by Z′OX′, X′OY′ have the same common point of intersection as the original rectangular co-ordinate planes ZOX, ZOY, XOY, and be inclined to these in any manner; then, if a given plane figure, in space, be projected upon any one of the first three planes as Z′OY′,

and the same figure be projected upon each of the three original co-ordinate planes, on again projecting each of these three projections on the plane Z′OY′, the sum of the three last projections will be equivalent to the area first projected on the same plane.

Imagine a line, designated OP to be drawn from the origin of the co-ordinates perpendicularly on the given plane figure which represent by A; then OX′ being perpendicular to Z′OY′, we have A cos. X′OP for the equivalent of the first projected area. Now (Art. 150.),

$$\cos. X'OP = \cos. XOP \ \cos. XOX' + \cos. YOP \ \cos. YOX' + \cos. ZOP \ \cos. ZOX':$$

therefore, multiplying each term by A,

$$A \cos. X'OP = A \cos. XOP \ \cos. XOX' + A \cos. YOP \ \cos. YOX' + A \cos. ZOP \ \cos. ZOX':$$

but A cos. XOP, which represent by A′, is equivalent to the given area projected on ZOY;

A cos. YOP $(= A'')$ is the area projected on ZOX,

and A cos. ZOP $(= A''')$ is the area projected on XOY.
Therefore, substituting, we get

$$A \cos. X'OP = A' \cos. XOX' + A'' \cos. YOX' + A''' \cos. ZOX'.$$

The terms in the second member of this equation are the projections of A′, A″, A‴ on Z′OY′; and thus their sum is proved to be equivalent to the first projected area on the same plane.

PROPOSITION IV.

170. Let any number of plane figures, in space, be projected on three rectangular co-ordinate planes ZOX, ZOY, XOY, and let the same figures be also projected on three other rectangular planes Z′OX′, Z′OY′, X′OY′ having the same point of common intersection as the former, but being inclined to them at any angles; it is required to prove that the sum of the squares of all the areas projected on the first system of co-ordinate planes, is equal to the sum of the squares of all the areas projected on the second system.

Let a, a', a'', &c. represent the areas of any number of plane figures, and imagine lines as OP, OP′, OP″, &c., to be let fall from O upon those planes; then

$$a \cos. XOP + a' \cos. XOP' + a'' \cos. XOP'' + \&c.$$

will be the sum of the projections on ZOY. Let this sum be represented by A', and let the sums of the projections of a, a', a'', &c. on ZOX, XOY be represented respectively by A'' and A'''; also let the sums of the projections of a, a', a'', &c., on $Z'OY'$, $Z'OX'$, $X'OY'$ be represented respectively by $A_{\prime}$, $A_{\prime\prime}$, $A_{\prime\prime\prime}$. Then, from the last proposition we have

$$A_{\prime} = A' \cos. XOX' + A'' \cos. YOX' + A''' \cos. ZOX' \quad . \quad . \quad (1)$$

$$A_{\prime\prime} = A' \cos. XOY' + A'' \cos. YOY' + A''' \cos. ZOY' \quad . \quad . \quad (2)$$

$$A_{\prime\prime\prime} = A' \cos. XOZ' + A'' \cos. YOZ' + A''' \cos. ZOZ' \quad . \quad . \quad (3)$$

On squaring both members of each of these equations, and adding the results together, it will be found that the sum of the coefficients of A'^2 is the sum of the squares of the cosines of the angles which OX makes with the three rectangular axes OX′, OY′, OZ′; therefore that sum will be equivalent to unity. In like manner it will be found that the sums of the coefficients of A''^2 and A'''^2 are the sums of the squares of the cosines of the angles which OY and OZ make with the same axes; and therefore those sums will be severally equal to unity. Again, the sum of the coefficients of $2A'A''$ is the sum of the products of the cosines of the angles which OX and OY make with OX′, OY′, OZ′; therefore, since OX and OY are at right angles to one another, the sum of those products (=cos. XOY) is zero. In like manner the sums of the coefficients of $2A'A'''$ and $2A''A'''$ are separately zero. Hence

$$A_{\prime}^2 + A_{\prime\prime}^2 + A_{\prime\prime\prime}^2 = A'^2 + A''^2 + A'''^2.$$

And if the co-ordinate planes are so situated that

$$A_{\prime\prime}^2 + A_{\prime\prime\prime}^2 = 0,$$

we should have

$$A_{\prime}^2 = A'^2 + A''^2 + A'''^2,$$

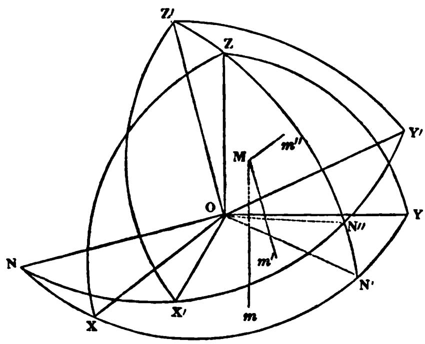
or the sum of the projections on Z′OY′ would be a maximum.

CHAP. IV.

TRANSFORMATION OF CO-ORDINATES.

PROPOSITION I.

171. To transform the co-ordinates of a point in space from one system of axes to another, both systems being rectangular, and having a common origin.

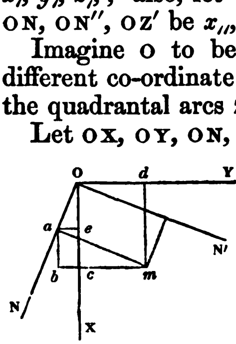


Let ZOX, ZOY, XOY be the original co-ordinate planes at right angles to one another, and let Z′OX′, Z′OY′, X′OY′ be the new planes, at right angles to one another; also let the planes YOX, Y′OX′, produced if necessary, intersect one another in ON.

Let M be any point in space; and let its co-ordinates with respect to OX, OY, OZ be x, y, z, while its co-ordinates with respect to OX′, OY′, OZ′ are x', y', z'. Imagine ON′ and ON″ to be drawn in the planes XOY, X′OY′, respectively perpendicular to ON, that is, in a plane passing through OZ and OZ′; and let the co-ordinates of M with respect to ON, ON′, OZ be $x_{,}$, $y_{,}$, $z_{,}$; also, let the co-ordinates of M with respect to ON, ON″, OZ′ be $x_{,,}$, $y_{,,}$, $z_{,,}$.

Imagine O to be the centre of a sphere, and let the different co-ordinate planes be those of great circles; also, let the quadrantal arcs ZX, ZY, &c. be drawn as in the figure.

Let OX, OY, ON, ON′ in the above figure be represented as in that which is annexed, and let m be the projection of M on XOY, also draw the perpendiculars to the co-ordinate axes as they are here represented. Then



Oe ($=Oa$ cos. NOX) $= x_{,}$ cos. NOX,

ab or ec ($= am$ sin. $amb = y_{,}$ sin. NOX;

therefore

$$x = x_{,} \cos. \text{NOX} + y_{,} \sin. \text{NOX} \ (= \text{O}c).$$

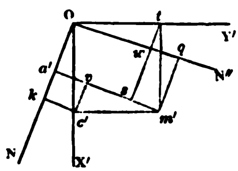
Also $$bm \ (= am \cos. amb) = y_{,} \cos. \text{NOX}$$

$$ae \text{ or } bc \ (= \text{O}a \sin. \text{NOX}) = x_{,} \sin. \text{NOX};$$

therefore $y = y_{,} \cos. \text{NOX} - x_{,} \sin. \text{NOX} \ (= cm \text{ or } \text{O}d)$

and $z = z_{,} \ (= \text{M}m)$ in the above figure.

Next, let OX′, OY′, ON, ON″ in the first figure be represented as in that which is annexed, and let m' be the projection of M on X′OY′; also draw the perpendiculars to the co-ordinate axes as they are here represented. Then



$$\text{O}k \ (= \text{O}c' \cos. \text{NOX}') = x' \cos. \text{NOX}',$$

$$c'v \text{ or } ka' (= c'm' \sin. c'm'v) = y' \sin. \text{NOX}';$$

therefore

$$x_{,,} = x' \cos. \text{NOX}' - y' \sin. \text{NOX}' \ (= \text{O}a')$$

Also

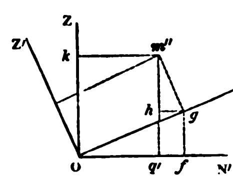
$$\text{O}w \ (= \text{O}t \cos. \text{Y}'\text{ON}'') = y' \cos. \text{NOX}',$$

$$m's \ (= m't \sin. m'ts) = x' \sin. \text{NOX}';$$

therefore $y_{,,} = x' \sin. \text{NOX}' + y' \cos. \text{NOX}' \ (= \text{O}q)$

and $z_{,,} = z' \ (= \text{M}m')$ in the first figure.

Lastly, let OZ′, ON′, ON″ be represented as in the annexed figure, and draw the perpendiculars to the co-ordinate axes; also, let m'' be the projection of M on the plane Z′ON′. Then



$$\text{O}f (= \text{O}g \cos. \text{N}'\text{ON}'') = y_{,,} \cos. \text{N}'\text{ON}'',$$

$$gh \text{ or } q'f \ (= m''g \sin. gm''k) = z_{,,} \sin. \text{N}'\text{ON}'';$$

therefore

$$y_{,} = y_{,,} \cos. \text{N}'\text{ON}'' - z_{,,} \sin. \text{N}'\text{ON}'' \ (= \text{O}q').$$

Also

$$m''h \ (= m''g \cos. gm''h) = z_{,,} \cos. \text{N}'\text{ON}'',$$

$$gf \text{ or } hq' \ (= \text{O}g \sin. \text{N}'\text{ON}'') = y_{,,} \sin. \text{N}'\text{ON}'';$$

therefore $z_{,} = z_{,,} \cos. \text{N}'\text{ON}'' + y_{,,} \sin. \text{N}'\text{ON}'' \ (= \text{O}k),$

and $x_{,} = x_{,,}.$

172. In the above values of $x_{,}, y_{,}, z_{,}$ substituting the equivalents of $x_{,,}, y_{,,}, z_{,,}$ from the next preceding equations, and in the values of x, y, z substituting the resulting values of $x_{,}$, $y_{,}$, $z_{,}$, there will be obtained, on representing the angle

N′ON″ by θ, NOX by ψ, and NOX′ by φ, the following equations:

$$x=x' \cos.\phi \ \cos.\psi-y' \sin.\phi \ \cos.\psi+(x' \sin.\phi+y' \cos.\phi)\cos.\theta \ \sin.\psi \\ -z' \sin.\theta \ \sin.\psi,$$

$$y=(x' \sin.\phi \cos.\theta+y' \cos.\varphi \cos.\theta-z' \sin.\theta) \cos.\psi \\ -x' \cos.\varphi \sin.\psi+y' \sin.\varphi \sin.\psi,$$

$$z=x' \sin.\varphi \sin.\theta+y' \cos.\varphi \sin.\theta+z' \cos.\theta.$$

173. Multiplying both members of each of these equations by the sum of the coefficients of x' in its second member, and adding the results together, the sum of the second members will become x', and we shall have

$$x'=x(\cos.\phi \ \cos.\psi+\sin.\phi \ \cos.\theta \ \sin.\psi)+y(\sin.\phi \ \cos.\theta \ \cos.\psi-\cos.\phi \ \sin.\psi) \\ +z' \sin.\phi \ \sin.\theta.$$

Multiplying both members of each of the equations for x, y, z by the sum of the coefficients of y' in its second member, and adding the results, we get

$$y'=x(\cos.\phi \ \cos.\theta \ \sin.\psi-\sin.\phi \ \cos.\psi)+y(\cos.\phi \ \cos.\theta \ \cos.\psi+\sin.\phi \ \sin.\psi) \\ +z \cos.\phi \ \sin.\theta.$$

In like manner, multiplying both members of each of the equations for x, y, z by the sum of the coefficients of z' in its second member, and adding the results, we have

$$z'=-x \ \sin.\theta \ \sin.\psi-y \ \sin.\theta \ \cos.\psi+z \ \cos.\theta.$$

174. When $\theta=0$, or the plane X′OY′ coincides with XOY, the values of x, y, z, and the corresponding values of x', y', z', will be more simple, since then $\sin.\theta=0$, and $\cos.\theta=1$. They will also be more simple if ON be supposed to coincide with OX, in which case NOX$=0$; whence $\sin.\psi=0$, and $\cos.\psi=1$. If both ON and OX′ be supposed to coincide with OX, we shall have $\psi=0$, and $\varphi=0$. In this last case

$$x=x',$$
$$y=y' \cos.\theta-z' \sin.\theta,$$
$$z=y' \sin.\theta+z' \cos.\theta.$$

175. In any system of rectangular co-ordinate planes, as ZOX, ZOY, XOY, if OM be represented by r, the angle MOm by μ, and the angle XOm by ν, we shall have

$$x=r \cos.\mu \cos.\nu,$$
$$y=r \cos.\mu \sin.\nu,$$
$$z=r \sin.\mu,$$

which values may be substituted in any equation relating to

x, y, z in rectangular co-ordinates, in order to convert it into an equation with polar co-ordinates.

176. If the system OX′, OY′, OZ′ were to move parallel to itself, so that the point of intersection is no longer coincident with the intersection O of the other system; representing this last intersection still by O, and the point of intersection in the translated system by A, the co-ordinates of A with respect to OX, OY, OZ being designated a, b, c, the values of x, y, z would differ from those given in Arts. 172. 174. 175. only in having a added to the equivalent of x, b to that of y, and c to that of z.

CHAP. V.

EQUATIONS FOR THE SURFACES OF SOLIDS PRODUCED BY REVOLUTION.

177. CURVE surfaces are supposed to be formed by the movement of a line, straight or curved, upon some other line, which may be also straight or curved. If the line in motion turn upon some fixed line as an axis, the figure produced by the movement is called a figure of revolution.

Thus, a curve line forming half the circumference of a circle, by revolving on the diameter produces the surface of a sphere. A straight line revolving parallel to itself about a fixed line, and at equal distances from it (measured perpendicularly to both lines), produces the surface of a cylinder; and a straight line revolving about a fixed line, so as to make everywhere equal angles with it, produces the surface of a cone. If the cylinder terminate above and below in a plane perpendicular to the fixed and revolving lines, and if the cone terminate below in a plane perpendicular to the fixed line, the cylinder and cone are said to be upright, and the bases are circles. If the fixed line in a cylinder or cone make with the base any angle less than a right angle, the solids formed by the revolution are said to be oblique.

Again, a straight line may revolve parallel to itself about a fixed line as an axis, at distances from it which, in perpendicular directions, are variable, and the surface formed by such revolving line is still said to be cylindrical; also, a straight line may revolve about an axis with which it makes an angle continually varying, and the surface so formed is still said to be conical.

178. DEF. 20. If half an ellipse, an hyperbola, or a parabola, revolve about one of the axes of the two first, or the axis of the last, the figure produced is called a spheroid, an hyperboloid, or a paraboloid of revolution.

The equation for the surface of a solid expresses the relations among the co-ordinates of any point on the surface with respect to the co-ordinate axes; and, in the following articles, the co-ordinates are supposed to be at right angles to one another.

PROPOSITION I.

179. To find the equation for a sphere.

If the origin of the co-ordinates be at the centre of the sphere whose radius is represented by r; x, y, x being the co-ordinates of any point in the surface, the equation is evidently

$$r^2 = x^2 + y^2 + z^2. \qquad (a)$$

If the origin be not at the centre, let α, β, γ be the co-ordinates of the centre; then the equation will be

$$r^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2: \qquad (b)$$

hence, if the origin be at any point on the surface of the sphere; since then

$$r^2 = \alpha^2 + \beta^2 + \gamma^2,$$

on developing (b) we have for the equation,

$$x^2 + y^2 + z^2 - 2\,(\alpha x + \beta y + \gamma z) = 0. \qquad (c)$$

PROPOSITION II.

180. To find the curve formed by the intersection of a plane with a sphere. (Fig. to Art. 171.)

For simplicity let the origin of the co-ordinates be at the centre of the sphere, so that the equation for the latter is (a) above; and let the cutting plane be parallel to X′OY′, passing through a given point A, whose co-ordinates on OX, OY, OZ are a, b, c. Imagine the co-ordinate axes OX′, OY′, OZ′, and also the former system of axes to be moved parallel to themselves, till O coincides with the given point A in the cutting plane. Imagine, in these new positions of the axes, that the plane X′AY′ intersects XAY in a line as AN, with which for simplicity let AX′ coincide. Then NAX′ $(\phi) = 0$, NAX $= \psi$, and the inclination of X′AY′ to XAY $= \theta$: also, x', y', z' being co-ordinates, with respect to Z′AX′, Z′AY′, X′AY′, of a point in the curve line formed by the intersection of the plane with the sphere, we shall have $z' = 0$.

Now the values of x, y, z, the co-ordinates of the same point with respect to ZOX, ZOY, XOY, the origin O having its former position at the centre of the sphere, will be (Arts. 172. 176.) since $\phi = 0$ and $z' = 0$,

$$x = a + x' \cos.\psi + y' \sin.\psi \cos.\theta$$
$$y = b - x' \sin.\psi + y' \cos.\psi \cos.\theta$$
$$z = c + y' \sin.\theta.$$

These values being substituted in the equation (a) Art. 179., there will be obtained an equation of the form

$$x'^2+y'^2+\text{A}y'+\text{B}x'=\text{C};$$

which being (Art. 54.) the equation for a circle, proves that every section of a sphere is a circle.

The proposition is manifest from Geometry (Geom. Cylind. Prop. I.).

PROPOSITION III.

181. To find the equation for a plane touching a sphere at a point on its surface, the origin of the co-ordinates being at the centre.

The equation for any plane passing through a given point in space is (Art. 165.)

$$\text{A}(x'-x)+\text{B}(y'-y)+\text{C}(z'-z)=0,$$

in which let x', y', z' be the co-ordinates of the point of contact.

The coefficients A, B, C denote the cosines of the angles which a perpendicular to the plane makes with the axes OX, OY, OZ, and this perpendicular passes through the centre of the sphere; therefore, r being the radius of the sphere,

$$\text{A}=\frac{x'}{r},\ \text{B}=\frac{y'}{r},\ \text{C}=\frac{z'}{r}:$$

substituting these values in the above equation for a plane, we get, since $x'^2+y'^2+z'^2=r^2$,

$$r^2=xx'+yy'+zz',$$

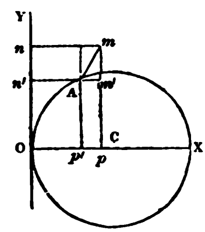
which is the required equation.

PROPOSITION IV.

182. To find the equation for a cylindrical surface the co-ordinates being rectangular.

Let the base of the cylinder be a circle, as OAX, in the plane of the paper, and let OX, OY be two co-ordinate axes in the same plane; then OZ (not shown in the figure) will be perpendicular to the paper; let OX be in the direction of a diameter of the circle, O the origin of the co-ordinates being on its circumference.

Let the axis of the cylinder be oblique to the base and consequently to the plane of the paper, and let the generating line meet the circumference of the base in some point as A, also from any point in that line let fall a perpendicular to the plane of the base, meeting that plane in m, and let the perpendicular be represented by z. Draw mp, Ap' perpendicular to OX, and mn, An' perpendicular to OY. Then, if a represent the tangent of the angle which, on ZOX, the projection of the generating line makes with OZ, and b the tangent of the angle which, on ZOY, the projection of the same line makes with OZ; also the co-ordinates Op, pm of the point above mentioned being x and y, and the co-ordinates Op', p'A of the point A in the same generating line being $x_{,}$ and $y_{,}$, we have



$$az=pp',\ bz=nn';$$

in which, since the generating line is supposed to move parallel to itself, the tangents a and b are invariable:

hence $$x_{,}=x-az, \text{ and } y_{,}=y-bz.$$

If the generating line be in a plane parallel to ZOX, Am will coincide with Am', or $b=0$; in which case $y_{,}=y$, and these equations become

$$x_{,}=x-az, \text{ and } y_{,}=y.$$

But, r being the radius of the base, the equation of the circle for any point as A is

$$2rx_{,}-x_{,}^2=y_{,}^2, \text{ or } 2rx_{,}=x_{,}^2+y_{,}^2.$$

Substituting the latter values of $x_{,}$ and $y_{,}$ in this last equation, we get, for the equation of the cylindrical surface when the generating line is parallel to the plane ZOX,

$$2r(x-az)=(x-az)^2+y^2. \qquad (b)$$

If the generating line were perpendicular to the base, a and b would vanish, and the equation for the cylinder would be

$$2rx=x^2+y^2,$$

the same as the equation for its base.

PROPOSITION V.

183. To find the intersection of a plane with an oblique cylinder.

Since the plane may have any position let it cut the circle OAX (Fig. to Prop. IV.) in OX, and be inclined to that circle in an angle expressed by θ. Let a point on the curve line in which the plane cuts the cylinder, have for its co-ordinates with respect to the three axes, x, y, z; and, in the cutting plane, let the co-ordinates of the same point be x', y', the third co-ordinate z' being zero.

Then the plane of the base and the cutting plane correspond to XOY, X′OY′ in the figure for the transformation of co-ordinates (Art. 171.); and, if it be supposed in that figure that ON and OX′ coincide with OX, ϕ and ψ being zero, also z' being zero, we have (Art. 174.)

$$x=x',\ y=y' \cos.\ \theta,\ \text{and}\ z=y' \sin.\ \theta.$$

Substituting these in the general equation [(a) Art. 182.] for the cylindrical surface, the latter becomes

$$2r\,(x'-ay' \sin.\ \theta)=(x'-ay' \sin.\ \theta)^2+y'^2 \cos.^2\ \theta,$$

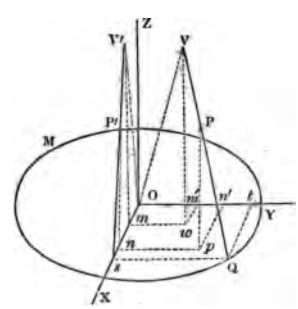
which (since x'^2 and y'^2 have like signs) is (Art. 48.) the equation for an ellipse, and proves the section to be such a curve.

The like conclusion would be obtained if the cylinder were upright.

PROPOSITION VI.

184. To find the equation for a cone having a circular base, the origin of the co-ordinates being at the centre of the base.

Let V be the vertex, MXQ the base, and VO the axis of an oblique cone: also let P be any point on the surface of the cone, and, through P, let a line be drawn from V to the circumference of the base. Imagine Vw, Pp to be let fall perpendicularly on the plane of the base, meeting it in w and p; and the lines wm, pn, Qs, and wm', pn', Qt, to be drawn perpendicularly to OX and OY. Again, imagine the triangle VOQ with the lines Vw and Pp to be projected on ZOX, and let



V′O*s* represent the projection of the triangle, V′*m* and P′*n* perpendicular to O*s* being the projections of V*w* and P*p*.

Let V′$m=h$, O$m=\alpha$, O$m'=\beta$; also let O$n=x$, O$n'=y$, P′$n=z$, O$s=x'$ and O$t=y'$; then

$$h-z : x-\alpha :: h : ms=\frac{(x-\alpha)\,h}{h-z};$$

consequently

$$x' \left(=\alpha+\frac{(x-\alpha)\,h}{h-z}\right)=\frac{hx-\alpha z}{h-z}.$$

In like manner, imagining the triangle OVQ to be projected on ZOY,

$$h-z : y-\beta :: h : m't=\frac{(y-\beta)\,h}{h-z};$$

and we get

$$y' \left(=\beta+\frac{(y-\beta)\,h}{h-z}\right)=\frac{hy-\beta z}{h-z}.$$

Substituting these values of x' and y' in the equation

$$x'^2+y'^2=r^2$$

for the circular base of the cone, the latter becomes

$$(hx-\alpha z)^2+(hy-\beta z)^2=r^2(h-z)^2,$$

which is the equation for any point in the surface of the cone.

Or, if OV be supposed to lie in the plane ZOY, VQ passing through P in any other plane, as MVQ; then $\alpha=0$, and the equation becomes

$$h^2x^2+(hy-\beta z)^2=r^2(h-z)^2. \qquad (a)$$

When the cone is upright, V*w* coincides in direction with OZ; and since then $\alpha=0$, $\beta=0$, we should have

$$h^2x^2+h^2y^2=r^2(h-z)^2 \text{ or } x^2+y^2=\frac{r^2}{h^2}(h-z)^2 \qquad (b)$$

for the equation of an upright cone having a circular base.

For an upright cone having an elliptical base:—Let the origin of the co-ordinates be at the centre, and let OX, OY coincide with the semi-transverse and semi-conjugate axes (t and c): also let the co-ordinates of P and of Q be represented as before. Then α and β being each zero, the above values of x' and y' become

$$\frac{hx}{h-z} \text{ and } \frac{hy}{h-z}:$$

substituting these values in $t^2y'^2+c^2x'^2=t^2c^2$, the equation [(b) Art. 56.] for the elliptical base, the latter becomes,

$$t^2h^2x^2+c^2h^2y^2=t^2c^2(h-z)^2 \text{ or } t^2x^2+c^2y^2=\frac{t^2c^2}{h^2}(h-z)^2, \quad (c)$$

which is the equation for such a cone.

If the origin of the co-ordinates be at V the vertex of an upright cone, r being the radius of the base if circular, or t and c the semi-axes of the base if elliptical; and h being the height, as before; then, z being measured from V, on putting z for $h-x$ in the second members of (b) and (c), the equation for the upright cone will become,
when the base is circular,

$$x^2+y^2=\frac{r^2}{h^2}z^2, \quad (b')$$

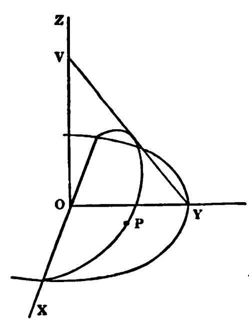
and when the base is elliptical,

$$t^2x^2+c^2y^2=\frac{t^2c^2}{h^2}z^2. \quad (c')$$

PROPOSITION VII.

185. To find the equation for the curve line formed by the intersection of a plane with the convex surface of an upright cone having a circular base.

Let the cutting plane pass through OX, and be inclined to the base of the cone in an angle expressed by θ. Then, if P be a point in the curve line of intersection, its co-ordinates with respect to ZOX, ZOY, XOY may, as in Art. 180., be represented by x, y, z, and with respect to XOY′ by x' and y', z' being zero; therefore we shall have



$$x=x', \quad y=y' \cos.\theta, \text{ and } z=y' \sin.\theta.$$

These being substituted in the equation

$$x^2+y^2=\frac{r^2}{h^2}(h-z)^2 \qquad \text{(Art. 184.)}$$

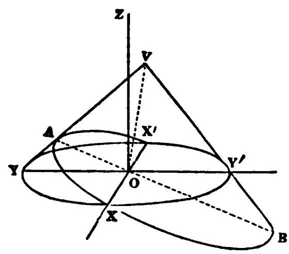
of the conical surface, the latter becomes

$$x'^2+y'^2\cos.^2\theta=\frac{r^2}{h^2}(h-y'\sin.\theta)^2,$$

or $\quad h^2x'^2+y'^2(h^2\cos.^2\theta-r^2\sin.^2\theta)+2r^2hy'\sin.\theta=r^2h^2$;
which is therefore an equation for the curve line.

If the coefficient of y'^2 is positive, the curve (Art. 48. *a.*) is an ellipse; if negative (Art. 48. *c.*), an hyperbola; and if zero (Arts. 46. and 48. *e.*), a parabola. Hence, if $h^2 \cos.^2 \theta$ is greater than $r^2 \sin.^2 \theta$, or $\tan. \theta$ is less than $\frac{h}{r}$; that is, if θ is less than the angle made by the side of the cone with the base, the section is an ellipse; if θ is greater than such angle, the section is an hyperbola; and if equal to it, the section is a parabola.

a. If an oblique cone have a circular base, and a section be made through OX (Fig. to Art. 185.); on substituting the above values of x, y, z in the equation (a) Art. 184., there would be obtained for the section an equation from which the section might be proved to be that of a circle when, as in the annexed figure, the angle YAO, which the cutting plane makes with VY, is equal to the supplement of the angle VY′O (the plane VY′Y being perpendicular to the base), which the base makes with the opposite side of the cone; but this may be more easily proved by the Elements of Geometry. For, let VY′ be produced till it meets AO produced in B; then the triangles YOA, Y′OB will be similar to one another; whence



$$\text{YO} : \text{OA} :: \text{OB} : \text{OY}', \text{ and } \text{OY} \,.\, \text{OY}' = \text{OA} \,.\, \text{OB}.$$

But the base YXY′ being a circle, OY . OY′ (Euc. 35. III.) $=\text{OX}^2$; therefore OA . OB $=\text{OX}^2$, which is a property of a circle; and consequently the section AXBX′ is a circle.

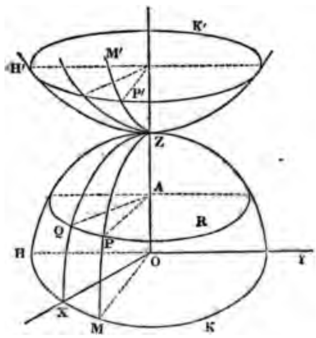
186. DEF. 21. A section formed in an oblique cone by a plane which makes the same angle with one side of the cone which the plane of the base makes with the opposite side, in a plane passing through the axis, is called a subcontrary section of the cone.

PROPOSITION VIII.

187. To find the equations for a spheroid, an hyperboloid, and a paraboloid.

Let the co-ordinate axes be rectangular, and let ZOX be part of an ellipse of which O is the centre; also, let, for example, OX be the semi-transverse ($=t$), and OZ the semi-

conjugate axis ($=c$); then, if any point, as Q, in the periphery revolve about OZ, it will evidently describe the circumference of a circle QR, whose centre is at A, where a perpendicular to OZ from Q meets that line. Here QR is a variable circle; and if its radius be represented by r, then r must be understood to vary with the place of that circle.



Let ZPM be any position of the revolving ellipse, and let the co-ordinates of P with respect to ZOX, ZOY, XOY be x, y, z; also, let the co-ordinates of P with respect to ZOM be z and r ($=$OA and AP); then the equation for P in the circle QR will be (Art. 56. *b.*)

$$x^2+y^2=r^2;$$

and in the ellipse [(a) Art. 56.]

$$\frac{t^2}{c^2}(c^2-z^2)=r^2, \text{ or } t^2c^2=c^2r^2+t^2z^2.$$

Substituting the first value of r^2 in the last equation, the latter becomes

$$t^2c^2=c^2(x^2+y^2)+t^2z^2,$$

which is the equation for any point P in the surface of the spheroid.

The equation for an hyperboloid, of which figure suppose H'ZK' to be a portion, is found in like manner, by substituting the first value of r^2 in the equation

$$t^2z^2-t^2c^2=c^2r^2 \qquad [(b) \text{ Art. } 57.]$$

for the hyperbola ZP'M'; from which we obtain for the required equation

$$t^2z^2-t^2c^2=c^2(x^2+y^2).$$

The equation for a paraboloid, of which figure suppose HZK to be a portion, is found by substituting the first value of r^2 in the equation $pz'=r^2$ (Art. 72.) for the parabola ZPM, p being put for the parameter of the axis ZO, and z' for ZA. Thus we obtain

$$pz'=x^2+y^2$$

for the equation of the paraboloid.

CHAP. VI.

CURVE SURFACES OF THE SECOND ORDER WHICH ARE NOT PRODUCED BY REVOLUTION.

188. THE most general equation for such surfaces is

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 + \text{D}xy + \text{E}xz + \text{F}yz + \text{G}x + \text{H}y + \text{K}z = \text{L}, \qquad (\text{A})$$

the co-ordinate axes being rectangular or oblique.

189. If the equations of any straight line intersecting a curve surface represented by the above equation be

$$x = az + b, \quad y = a'z + b',$$

on substituting the second members in place of x and y in that equation, the result will be a quadratic equation with respect to the variable quantity z; therefore there can be but two real values of that variable, and consequently also but two real values of x and y for the points in which the straight line intersects the curve. Thus a straight line can cut a curve surface of the second order in only two points. Such line is called a chord of the curve surface.

PROPOSITION I.

190. If a plane bisect any three parallel chords not in one plane, it will bisect every chord parallel to these.

Let the bisected portions of the three given chords be represented by $+k'$ and $-k'$, $+k''$ and $-k''$, $+k'''$ and $-k'''$, and let the co-ordinates of the middle points be $(\gamma', \delta', \epsilon')$, $(\gamma'', \delta'', \epsilon'')$, $(\gamma''', \delta''', \epsilon''')$; then the co-ordinates of the extremities of $+k'$, $+k''$, $+k'''$ in the curve surface may be represented by

$$\begin{array}{llll} & \gamma' + uk', & \delta' + vk', & \epsilon' + tk', \\ & \gamma'' + uk'', & \delta'' + vk'', & \epsilon'' + tk'', \\ \text{and} & \gamma''' + uk''', & \delta''' + vk''', & \epsilon''' + tk'''; \end{array}$$

in which u, v, t denote the ratios between the sines of the angles which k', k'', k''', and lines imagined to be drawn from their extremities and the extremities of their co-ordinates, with respect to the middle points, make with one another

and with the co-ordinates. These co-ordinates for each chord hold the places of x, y, z respectively in the general equation (A); therefore, on substituting them, that equation becomes, for the chord k',

$$\text{A}(\gamma'+uk')^2+\text{B}(\delta'+vk')^2+\text{C}(\epsilon'+tk')^2+\text{D}(\gamma'+uk')(\delta'+vk')+\text{E}(\gamma'+uk')(\epsilon'+tk') \\ +\text{F}(\delta'+vk')(\epsilon'+tk')+\text{G}(\gamma'+uk')+\text{H}(\delta'+vk')+\text{K}(\epsilon'+tk')=\text{L}. \qquad (a)$$

Imagining this to be resolved as a quadratic equation, the two roots $+k'$ and $-k'$ being equal to one another and having contrary signs, the sum of the coefficients of the first power of k' must be zero; therefore, collecting these coefficients, we have

$$2\text{A}u\gamma'+2\text{B}v\delta'+2\text{C}t\epsilon'+\text{D}v\gamma'+\text{D}u\delta'+\text{E}t\gamma'+\text{E}u\epsilon'+\text{F}t\delta'+\text{F}v\epsilon' \\ +\text{G}u+\text{H}v+\text{K}t=0;$$

or putting $p\epsilon'$ for γ', which may be done if a corresponding point be assumed as the origin of the co-ordinates,

$$2\text{A}pu\epsilon'+2\text{B}v\delta'+2\text{C}t\epsilon'+\text{D}pv\epsilon'+\text{D}u\delta'+\text{E}pt\epsilon'+\text{E}u\epsilon'+\text{F}t\delta'+\text{F}v\epsilon' \qquad (b) \\ +\text{G}u+\text{H}v+\text{K}t=0.$$

Putting first double, and then triple accents on δ and ϵ, corresponding equations will be obtained for the chords k'', k'''; therefore, subtracting successively the equation for k' from those for k'' and k''', and reducing,

$$\frac{\epsilon''-\epsilon'}{\delta''-\delta'}\left(=\frac{\epsilon'''-\epsilon'}{\delta'''-\delta'}\right)=-\frac{2\text{B}v+\text{D}u+\text{F}t}{2\text{A}pu+2\text{C}t+\text{D}pv+\text{E}pt+\text{E}u+\text{F}v}.$$

On substituting $q\epsilon'$ for δ', $q\epsilon''$ for δ'', and proceeding in like manner there would be obtained a value of $\frac{\epsilon''-\epsilon'}{\gamma''-\gamma'}$, or its equal $\frac{\epsilon'''-\epsilon'}{\gamma'''-\gamma'}$.

Now, imagine a fourth chord to be parallel to the three former, and let either of its segments made by the plane which bisects the others be represented by k; also let γ, δ, ϵ be the co-ordinates of the point in which it is intersected by that plane; then $\gamma+uk$, $\delta+vk$, $\epsilon+tk$ will be co-ordinates of either of the points in which that chord cuts the curve surface; and substituting these co-ordinates for x, y, z in the equation (A), the result will be identical with (a) if the accents in the latter be effaced. But all the points in which the four chords are intersected by the given plane being, of course, in that plane, and the co-ordinate lines being parallel to one another, the differences between the like co-ordinates of the four points are proportional; therefore

$$\frac{\epsilon''-\epsilon'}{\delta''-\delta'}=\frac{\epsilon'''-\epsilon'}{\delta'''-\delta'}=\frac{\epsilon'-\epsilon}{\delta'-\delta},$$

and $$\frac{\epsilon''-\epsilon'}{\gamma''-\gamma'}=\frac{\epsilon'''-\epsilon'}{\gamma'''-\gamma'}=\frac{\epsilon'-\epsilon}{\gamma'-\gamma};$$

we may also put $p\epsilon$ for γ or $q\epsilon$ for δ.

Substituting $p\epsilon$ for γ in the equation said to be identical with (a) when the accents in the latter are omitted, and resolving the equation as a quadratic with respect to k, the sum of the coefficients of the first power of k will have for its numerator

$$2\text{A}pu\epsilon+2\text{B}v\delta+2\text{C}t\epsilon+\text{D}pv\epsilon+\text{D}u\delta+\text{E}pt\epsilon+\text{E}u\epsilon+\text{F}t\delta+\text{F}v\epsilon \\ +\text{G}u+\text{H}v+\text{K}t, \qquad (b')$$

the denominator being the sum of the coefficients of k^2.

Next, substituting $\frac{\epsilon'-\epsilon}{\delta'-\delta}$ for $\frac{\epsilon''-\epsilon'}{\delta''-\delta'}$, we get

$$\frac{\epsilon'-\epsilon}{\delta'-\delta}=-\frac{2\text{B}v+\text{D}u+\text{F}t}{2\text{A}pu+2\text{C}t+\text{D}pv+\text{E}pt+\text{E}u+\text{F}v};$$

whence

$$2\text{A}pu\epsilon'+2\text{C}t\epsilon'+\text{D}pv\epsilon'+\text{E}pt\epsilon'+\text{E}u\epsilon'+\text{F}v\epsilon'+2\text{B}v\delta'+\text{D}u\delta'+\text{F}t\delta' \\ =2\text{A}pu\epsilon+2\text{C}t\epsilon+\text{D}pv\epsilon+\text{E}pt\epsilon+\text{E}u\epsilon+\text{F}v\epsilon+2\text{B}v\delta+\text{D}u\delta+\text{F}t\delta. \qquad (c)$$

But the first nine terms in (b') constitute the second member of the equation (c); they are therefore equivalent to its first member; and, comparing (b') with (b), it will be seen that the former, that is the sum of the coefficients of the first power of k, is zero, or the two values of k in the quadratic equation are equal to one another. Thus the fourth chord is bisected by the plane which bisects the three others; and it follows that the same plane will bisect all chords which are parallel to these.

The equation for this plane becomes, on making (b') equal to zero and restoring γ for $p\epsilon$,

$$(2\text{A}u+\text{D}v+\text{E}t)\gamma+(2\text{B}v+\text{D}u+\text{F}t)\delta+(2\text{C}t+\text{E}u+\text{F}v)\epsilon \\ +\text{G}u+\text{H}v+\text{K}t=0; \qquad (d)$$

or, dividing every term by t,

$$\left(2\text{A}\frac{u}{t}+\text{D}\frac{v}{t}+\text{E}\right)\gamma+\left(2\text{B}\frac{v}{t}+\text{D}\frac{u}{t}+\text{F}\right)\delta+\left(2\text{C}+\text{E}\frac{u}{t}+\text{F}\frac{v}{t}\right)\epsilon \\ +\text{G}\frac{u}{t}+\text{H}\frac{v}{t}+\text{K}=0. \qquad (d')$$

191. DEF. 22. A plane bisecting all the chords which are parallel to one another is called a diametral plane. The intersection of any two diametral planes is evidently a diameter of the curve surface.

192. COR. If, in the equation (d'), u and γ, v and δ, t and ϵ be respectively interchanged, we should have

$$\left(2A\frac{\gamma}{\epsilon}+D\frac{\delta}{\epsilon}+E\right)u+\left(2B\frac{\delta}{\epsilon}+D\frac{\gamma}{\epsilon}+F\right)v+\left(2C+E\frac{\gamma}{\epsilon}+F\frac{\delta}{\epsilon}\right)t$$
$$+G\frac{\gamma}{\epsilon}+H\frac{\delta}{\epsilon}+K=0:$$

this is the equation for a plane of which u, v, t represent the general co-ordinates, and which bisects all chords parallel to the plane whose general co-ordinates are γ, δ, ϵ. Thus the diametral plane which bisects chords parallel to another diametral plane, is parallel to the chords bisected by the latter.

Let the two planes be designated A and B, and let a third plane C which bisects chords parallel to B be so situated that these bisected chords are also parallel to A; then C will bisect the intersection of A and B. Now, since B will bisect chords parallel to C, and it already bisects chords parallel to A, it will bisect the intersection of A and C. In like manner it may be shown that A will bisect the intersection of B and C. Thus each of the three diametral planes will bisect the chords which are parallel to the intersection of the two others.

193. DEF. 23. Three planes so situated are called conjugate diametral planes; and it is manifest that, for a curve surface of the second order, there may be an infinite number of such conjugate diametral planes. The common intersection of all the diametral planes is the centre of the curve surface.

194. The equation for any diametral plane being of the form expressed in (d'); when one of the chords which it bisects coincides with the axis of x in the general equation (A), the angle in the denominator of the fraction which expresses the value of u becomes zero, in which case u or $\frac{u}{t}$ becomes infinite: again, when one of the chords is parallel to the axis of y, the angle in the denominator of v vanishes; whence v or $\frac{v}{t}$ becomes infinite: lastly, when one of the chords is parallel to the axis of z the angles in the numerators both of u and v vanish; whence both $\frac{u}{t}=0$ and $\frac{v}{t}=0$. Making therefore $\frac{u}{t}$ infinite in (d') and dividing every term by it; that equation is reduced to

$$2A\gamma+D\delta+E\epsilon+G=0:$$

again making $\frac{v}{t}$ infinite, and dividing by it,

$$D\gamma + 2B\delta + F\epsilon + H = 0;$$

and, making both $\frac{u}{t} = 0$ and $\frac{v}{t} = 0$, the equation becomes

$$E\gamma + F\delta + 2C\epsilon + K = 0.$$

These are the equations of three diametral planes which bisect, respectively, the chords which are parallel to the axes of x, y, z, or of γ, δ, ϵ. Now, let the co-ordinate planes be represented by ZOX, ZOY, XOY, and let them be parallel to three conjugate diametral planes; then the diametral plane represented by the first of these equations being parallel to ZOY, γ is constant, while δ and ϵ are variable; the second being parallel to ZOX, δ is constant while γ and ϵ are variable; and the third being parallel to XOY, ϵ is constant while γ and δ are variable. But every equation in which some terms are constant and others variable is absurd; and to render such equation consistent with itself, the coefficients of the variable quantities must be separately zero. Therefore, in the above equations, $D=0$, $E=0$, $F=0$. It follows that, when the co-ordinate planes are parallel to a system of conjugate diametral planes, the general equation (A) must have the form

$$Ax^2 + By^2 + Cz^2 + Gx + Hy + Kz = L.$$

If the co-ordinate planes be co-incident with any system of conjugate diametral planes, the constant terms G, H, K in the three equations above must be separately zero; therefore, in this case, the general equation (A) must have the form

$$Ax^2 + By^2 + Cz^2 = L. \qquad (A')$$

The general equation (A) might have been reduced to the form at (A′) by transformations of the co-ordinates of a point in the curve surface, and the establishment of equations of condition among the arbitrary quantities. (La Croix, Traité du Calcul Differentiel et Integral, No. 301.) In that work the values of the arbitrary quantities are proved to be real, and it is shown that there may be one system of conjugate diametral planes at right angles to one another; it is also shown that, when the curve surface is not one of revolution, there can be only one such system: the co-ordinates of the centre of the curve surface are investigated, and the conditions under which that centre may be infinitely distant from the surface are thence inferred.

195. DEF. 24. The intersections of three conjugate dia-

metral planes at right angles to one another are called the principal axes of the curve surface.

196. The equation (A′) may be expressed in terms of the conjugate diameters formed by the intersections of any system of conjugate diametral planes, rectangular or oblique:—thus, let a, b, c, be the conjugate semi-diameters; then, in that equation, on making $y=0$ and $z=0$, when $x=a$,

$$\text{A}a^2 = \text{L}, \text{ or } \text{A} = \frac{\text{L}}{a^2}:$$

also, making $x=0$, $z=0$; when $y=b$,

$$\text{B}b^2 = \text{L}, \text{ or } \text{B} = \frac{\text{L}}{b^2}:$$

again, making $x=0$, $y=0$; when $z=c$,

$$\text{C}c = \text{L}, \text{ or } \text{C} = \frac{\text{L}}{c^2}.$$

It follows that the above equation may be put in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The curve surfaces to which the equation just given relates will be of different kinds, according to the signs of the terms in its first member.

PROPOSITION II.

197. To find the nature of the curve resulting from the intersection of the figure by a plane parallel to one of the co-ordinate planes; all the terms in the equation for the surface being positive.

Note. In the following propositions of this section, the co-ordinate axes are supposed to coincide with the principal axes of the curve surface.

Let the co-ordinate plane be XOY; then z being a given or constant quantity, if it be represented by m the equation of the surface will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{m^2}{c^2} = 1, \text{ or } \frac{c^2 x^2}{a^2(c^2 - m^2)} + \frac{c^2 y^2}{b^2(c^2 - m^2)} = 1;$$

which, while m is less than c, is manifestly [(b) Art. 56.] the equation for an ellipse, and it follows that a section parallel to, or (supposing z or m to be zero) coincident with XOY, is an ellipse. In like manner it may be shown, that a section pa-

rallel to, or coincident with ZOX or ZOY, is an ellipse. If m were greater than c the section would be imaginary.

198. DEF. 25. A curve surface of this kind is called an ellipsoid; it becomes a spheroid (or figure formed by the revolution of a semi-ellipse about one of its principal diameters) when $a=b$, and a sphere when $a=b=c$.

a. The ellipsoid is bounded in every direction; since, if we represent the co-ordinates of a point in which any diameter cuts the curve surface by $x=\alpha z$, $y=\beta z$, and substitute these values in the equation of the ellipsoid, we get

$$z^2=\frac{a^2 b^2 c^2}{b^2 c^2 \alpha^2 + a^2 c^2 \beta^2 + a^2 b^2};$$

therefore z is always real, and consequently so are x and y.

The ellipsoid having thus but one surface is said to have one sheet.

PROPOSITION III.

199. To find the form of the curve produced when the surface is intersected by a plane parallel to one of the co-ordinate planes; the sign of z^2 in the equation for the surface being negative.

Let the plane be parallel to XOY, so that z may be constant; then, if $z=m$, the equation of the surface (Art. 196.) will become

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{m^2}{c^2}, \text{ or } \frac{c^2 x^2}{a^2(c^2+m^2)}+\frac{c^2 y^2}{b^2(c^2+m^2)}=1.$$

This is [(*b*) Art. 56.] the equation for an ellipse, and the sections made by a plane parallel to, and coincident (when $z=0$) with XOY, are ellipses: in the latter case the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If the cutting plane be parallel to ZOX, in which case y is constant, putting n for y, we get

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{n^2}{b^2};$$

and if the cutting plane be coincident with ZOX, so that $y=0$, we have

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1:$$

either of which equations appertains [(*b*) Art. 57.] to an hyperbola.

In like manner, if a section be parallel to ZOY, putting p for x, or coincident with ZOY, in which case $x=0$, we have

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{p^2}{a^2}, \text{ or } \frac{y^2}{b^2}-\frac{z^2}{c^2}=1:$$

either of which appertains also to an hyperbola.

200. In the section which coincides with YOX; if $x=0$, $y=\pm b$; also, if $y=0$, $x=\pm a$. In the section which coincides with ZOX, if $x=0$, z is imaginary: and in the section which coincides with ZOY, if $y=0$, z is imaginary.

If the equations for any diameter be $x=\alpha z$, $y=\beta z$; on substituting these in the equation for the curve surface (Art. 196.), we have (z^2 being negative), for the point in which the diameter meets the surface,

$$z^2=\frac{a^2 b^2 c^2}{b^2 c^2 \alpha^2+a^2 c^2 \beta^2-a^2 b^2},$$

in which the value of z is real only when the denominator is positive.

If the denominator were zero, z would be infinite, and the diameter would be an asymptote to the curve: on making the denominator zero we have

$$b^2 c^2 \alpha^2+a^2 c^2 \beta^2=a^2 b^2;$$

whence $\beta=\pm\frac{b}{ac}\sqrt{(a^2-\alpha^2 c^2)}$, and the value of β is real, while α is less than $\frac{a}{c}$.

Hence $x=\alpha z$ and $y=\pm\frac{bz}{ac}\sqrt{(a^2-\alpha^2 c^2)}$ are the equations for an asymptote to the curve surface.

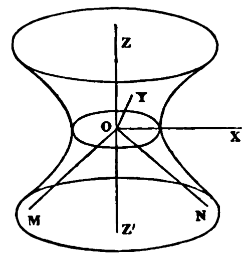
From these equations we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2},$$

which [(b') Art. 184.] is the equation for the surface of a cone; and it follows that the surface of a cone is contained within that of the figure (the centre of the latter being the vertex of the cone), and meets it at an infinite distance from O. Such cone is therefore an asymptote to the curve surface of the figure.

201. DEF. 26. The curve surface corresponding to the equation assumed in the proposition, that is, the equation in Art. 196., when z^2 is negative, is called an hyperboloid.

A section of the hyperboloid in the plane, XOY, and in every plane parallel to it, is an ellipse, while a section in every plane passing through OZ, or parallel to such plane, is an hyperbola; but the axis ZOZ′, if produced indefinitely, does not meet the surface: thus the hyperboloid is one continuous curve surface extending to infinity in the directions OZ and OZ′. It is therefore said to have one sheet. OM and ON are the directions of two asymptotes.



In the general equation of the hyperboloid, if $a=b$ we have

$$x^2+y^2-\frac{a^2}{c^2}z^2=a^2,$$

which is the equation for an hyperboloid of revolution.

PROPOSITION IV.

202. To find the nature of the curves resulting from the intersection of the figure by a plane parallel to one of the co-ordinate planes; when, in the equation for the surface, the signs of y^2 and z^2 are negative.

Let the cutting plane be parallel to XOY so that z may be represented by n a constant; then the general equation (Art. 196.) becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1+\frac{n^2}{c^2},$$

which [(b) Art. 57.] is the equation for an hyperbola consisting of two branches proceeding from the opposite extremities of the transverse axis.

Likewise, if the cutting plane be parallel to ZOX, so that y may be represented by p a constant, the equation becomes

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1+\frac{p^2}{b^2},$$

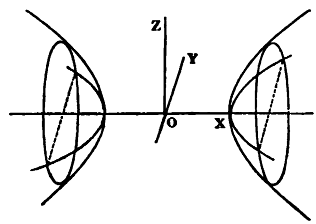
which is also that of an hyperbola consisting of two branches proceeding from the opposite extremities of the transverse axis. If the cutting plane were coincident with XOY or ZOX (when $n=0$ or $p=0$) the equations would still be those of hyperbolas.

If the cutting plane be parallel to ZOY so that x may be represented by q a constant, the equation (Art. 196.) becomes

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{q^2}{a^2}-1:$$

this is equation for an ellipse when $\frac{q^2}{a^2}$ is greater than 1, or q is greater than a. When q is less than a, or when $q=0$, the section being imaginary there are no branches in that plane, and OZ, being continued indefinitely either way, does not meet the curve.

203. DEF. 27. Since the sections of this curve surface are hyperbolas, in the planes XOY and ZOX, on opposite sides of O; and that there is no section in the plane ZOY, the curve surface is called an hyperboloid with two sheets.



204. By a process analogous to that by which the origin of the co-ordinates for a plane curve is transferred from the centre of the curve to one extremity of a diameter (Art. 58.), the co-ordinate of a curve surface may be so transferred.

For example, let the intersection O of the rectangular coordinate planes be at an extremity of the axis OX; then, while y and z may remain in the general equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1,$$

x may be changed into $a-x'$, so that x', y, z may become the co-ordinates of any point on the curve surface in these situations of the axes.

The equation thus becomes

$$\frac{a^2-2ax'+x'^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1, \text{ or } \frac{x'(x'-2a)}{a^2}+\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0.$$

PROPOSITION V.

205. To find the nature of the curve surface when its centre is infinitely remote.

In the last of the above equations a^2 may represent the product $a\times a'$ of the two abscissæ on the axis OX between the centre O of the curve and the extremities of the axis of the curve, also $2a$ may represent $a+a'$; therefore, if the curve have no centre, or its axis be infinite in this direction, one of the terms a or a' (suppose it be a) will be infinite: consequently, the first term in that equation being written in the form $\frac{x'(x-a-a')}{a\times a'}$, the factor $x-a-a'$, or $-(a+a'-x)$,

as well as a in the denominator, will be infinite; and the fraction will be equivalent to $\frac{x'}{a'}$. Thus the equation becomes, using the positive sign with z^2,

$$-\frac{x'}{a'}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0, \text{ or } \frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x'}{a'}$$

If the curve surface represented by this equation be cut by a plane parallel to the plane XOY, so that $\pm z$ may be represented by m a constant, the equation will become, suppressing the accents,

$$\frac{y^2}{b^2}=\frac{x}{a}-\frac{m^2}{c^2}, \quad \text{or } y^2=\frac{b^2}{a}\left(x-\frac{am^2}{c^2}\right),$$

which (Art. 63.) is the equation for a parabola. In like manner, if the curve be cut by a plane parallel to ZOX, so that y may be represented by n a constant, the equation will be

$$\frac{z^2}{c^2}=\frac{x}{a}-\frac{n^2}{b^2},$$

which is also the equation of a parabola.

If the curve surface be cut by a plane parallel to ZOY, so that x may be represented by p a constant, the equation will be

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{p}{a},$$

which is the equation for an ellipse.

206. DEF. 28. A curve surface of this kind is called an elliptical paraboloid.

If the negative sign be used with z^2 in the equation (Art. 205.), it will be found that the sections parallel to, or coincident with XOY, ZOX are parabolas; and that the section parallel to, or coincident with ZOY is an hyperbola.

207. DEF. 29. Such a curve surface is called an hyperbolic paraboloid.

PROPOSITION VI.

208. To find the nature of the curve obtained by the intersection of a plane with an ellipsoid; the plane having any position.

Let the co-ordinates of a point in the line of intersection, with respect to rectangular co-ordinate axes ZOX, ZOY, XOY (Fig. to Art. 171.), supposed to coincide with the principal axes of the figure, be x, y, z; and first let the plane of section

be X′OY′, passing through O the centre of the curve surface. Let OX′ coincide with the intersection of X′OY′ with XOY, so that NOX′, or ϕ, $=0$, NOX or X′OX$=\psi$, and let the inclination of the plane X′OY′ to XOY be θ: also, in the plane of the section, let x' and y' be rectangular co-ordinates of the point in the curve line, z' being zero. Then (Art. 172.)

$$x=x' \cos.\psi+y' \cos.\theta \sin.\psi$$
$$y=-x' \sin.\psi+y' \cos.\theta \cos.\psi$$
$$z=y' \sin.\theta:$$

these values being substituted in the equation of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the latter will become $1=$

$$\frac{1}{a^2}(x'^2 \cos.^2\psi+2\,x'y' \cos.\theta \sin.\psi \cos.\psi+y'^2 \cos.^2\theta \sin.^2\psi)$$
$$+\ \frac{1}{b^2}(x'^2 \sin.^2\psi-2\,x'y' \cos.\theta \sin.\psi \cos.\psi+y'^2 \cos.^2\theta \cos.^2\psi)$$
$$+\ \frac{1}{c^2}y'^2 \sin.^2\theta;$$

or, arranging the terms, $1=$

$$\left(\frac{\cos.^2\psi}{a^2}+\frac{\sin.^2\psi}{b^2}\right)x'^2+\left(\frac{\cos.^2\theta \sin.^2\psi}{a^2}+\frac{\cos.^2\theta \cos.^2\psi}{b^2}+\frac{\sin.^2\theta}{c^2}\right)y'^2$$
$$+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)2\,x'y' \cos.\theta \sin.\psi \cos.\psi.$$

This is an equation of the second degree; now representing the co-efficient of y'^2 by A or B, that of x'^2 by B or A, and that of $x'y'$ by C, it will be found that C^2-4AB is negative, which (Art. 48. and 48 *a*.) is characteristic of an ellipse; therefore a section passing in any position through the centre of the curve surface is an ellipse.

If the plane of the section do not pass through the centre of the curve surface, let the origin of the co-ordinates x', y' in the plane of the section have, with respect to the centre, that is to O, co-ordinates represented by α, β, γ. Then we shall have

$$x=\alpha+x' \cos.\psi+y' \cos.\theta \sin.\psi$$
$$y=\beta-x' \sin.\psi+y' \cos.\theta \cos.\psi$$
$$z=\gamma+y' \sin.\theta;$$

and substituting as before, it will be found that the resulting equation is that of an ellipse: thus, whatever be the position of the cutting plane, the section is an ellipse.

a. **By similar processes it may be shown that a section of an hyperboloid, either of one or two sheets, is an ellipse or an hyperbola; that a section of an elliptic paraboloid is an ellipse or a parabola, and that a section of an hyperbolic paraboloid is an hyperbola or a parabola.**

PROPOSITION VII.

209. To find the equation for a plane touching a curve surface.

Let a straight line be supposed to cut the curve surface in two points, whose co-ordinates are (x', y', z'), (x'', y'', z''): then the equation of the surface will be, for those points respectively, [(A') Art. 194.]

$$\text{A}x'^2+\text{B}y'^2+\text{C}z'^2=\text{L}, \text{ and } \text{A}x''^2+\text{B}y''^2+\text{C}z''^2=\text{L};$$

therefore, by subtraction,

$$\text{A}\,(x'^2-x''^2)+\text{B}\,(y'^2-y''^2)+\text{C}\,(z'^2-z''^2)=0,$$

or $$\text{A}\,(x'-x'')\,(x'+x'')+\text{B}\,(y'-y'')\,(y'+y'')+\text{C}\,(z'-z'')\,(z'+z'')=0.$$

But the equations for a straight line passing through those points are (Art. 146.),

$$x'-x''=\alpha\,(z'-z'') \text{ and } y'-y''=\beta\,(z'-z''):$$

therefore, substituting these values of $x'-x''$ and $y'-y''$, and dividing by $z'-z''$, the equation becomes

$$\text{A}\alpha\,(x'+x'')+\text{B}\beta\,(y'+y'')+\text{C}\,(z'+z'')=0;$$

which, when the line becomes a tangent, so that $x''=x'$, $y''=y'$, $z''=z'$, becomes

$$\text{A}\alpha x'+\text{B}\beta y'+\text{C}z'=0.$$

Now, if a plane meet a curve surface in one point, every line drawn in that plane through the point of contact will be a tangent to the surface; therefore (x', y', z') being co-ordinates of the point of contact, and (x, y, z) co-ordinates of any point in the tangent plane, the equations for a line passing through that point are

$$x'-x=\alpha\,(z'-z); \text{ whence } \alpha=\frac{x'-x}{z'-z},$$

and $$y'-y=\beta\,(z'-z); \text{ whence } \beta=\frac{y'-y}{z'-z}:$$

consequently, by substitution, the equation for the tangent plane becomes

$$\text{A}\,\frac{x'-x}{z'-z}\,x'+\text{B}\,\frac{y'-y}{z'-z}\,y'+\text{C}z'=0;$$

whence $\text{A}x'^2 - \text{A}x'x + \text{B}y'^2 - \text{B}y'y + \text{C}z'^2 - \text{C}z'z = 0$,

or $\quad \text{A}x'^2 + \text{B}y'^2 + \text{C}z'^2 = \text{A}x'x + \text{B}y'y + \text{C}z'z$:

that is

$$\text{A}x'x + \text{B}y'y + \text{C}z'z = \text{L}, \text{ or (Art. 196.) } \frac{x'x}{a^2} + \frac{y'y}{b^2} + \frac{z'z}{c^2} = 1.$$

The required equation for the tangent plane; and, when all the terms are positive, it is the equation for the tangent plane to an ellipsoid.

210. COR. Since $\text{A}x'$, $\text{B}y'$, $\text{C}z'$, here hold the places of A, B, C, in the general equation [(d'') Art. 152.] for any plane, it follows, from (g) Art. 160, that the equations for a normal to the curve surface are

$$x' - x_{,} = \frac{\text{A}x'}{\text{C}z'}(z' - z_{,}), \text{ and } y' - y_{,} = \frac{\text{B}y'}{\text{C}z'}(z' - z_{,});$$

x', y', z', being the co-ordinates of the point at which the normal meets the curve, and $x_{,}$, $y_{,}$, $z_{,}$, the co-ordinates of any other point in the normal line.

APPENDIX.

ELEMENTARY PROPOSITIONS RELATING TO DESCRIPTIVE GEOMETRY.

211. In this branch of science points or lines in space are supposed to be projected orthogonally on two rectangular co-ordinate planes, of which one is generally conceived to be in a horizontal and the other in a vertical position. The former of these, in the figures to the following propositions, is represented by YOX, and the other by ZOX, agreeably to the designations used in the analytical geometry of three dimensions: YOX is that which is properly in the plane of the paper, ZOX which must be understood to be at right angles to the paper, being, however, for convenience, represented as in the same plane; the line OX is that in which the two planes should intersect one another.

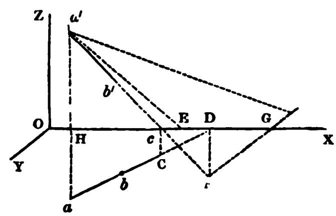
212. Instead of two co-ordinate planes, one only is sometimes employed, and, in this case, the lines or planes in space are supposed to be projected orthogonally on a plane, as YOX, which is conceived to be in a horizontal position: in place of the projections on ZOX, there are exhibited, in numbers, the heights of points in such lines or planes vertically above YOX, this last is then designated "a plane of comparison." In using the latter method of representing points, lines, or figures in space, two kinds of scales are necessary; one of these serves to measure the horizontal distances between points projected on the plane of comparison, and is no other than the usual scale of a topographical plan: the other shows, by the number adjacent to any one of its graduations, the vertical height, at that place, of the point in space of which the graduation is the horizontal projection. Such graduated line is therefore called a scale of heights.

PROPOSITION I.

213. To determine the intersection of a line given in space, with either of the co-ordinate planes, and the inclination of the line to that plane.

Let the given line be designated AB, A and B being any two points in it: also let a and b, a' and b' be the orthogonal projections of A and B, the two former points on the plane XOY, and the two latter on ZOX.

Join a and b, a' and b', and produce the lines till they meet OX, in D and c; also, from one of the points as c draw Cc perpendicular to OX, meeting ab produced in C; then C will be the point in which AB produced intersects the plane XOY.



For c on the plane XOY is in the projected line $a'b'$, and is therefore the projection on OX of the point in which the given line AB produced cuts the plane XOY; and since the given line AB (not being parallel to that plane) must intersect XOY somewhere in ab, produced if necessary, it follows that C, of which c is the projection, must be the point in which AB cuts XOY.

The angle at which the given line is inclined to the plane XOY may be found thus:—Draw aH, Ha' perpendicular to OX, and on OX make HE equal to aC; also join a'E: then, since the height of A above a is equal to Ha', and since AB meets ab produced in C, the angle HEa' is equal to the required inclination.

To make the scale of vertical heights for the line AB in space: let the heights of the line above a and b be, for example, 8 feet and 3 feet. Divide ab into five equal parts ($=8-3$), and continue the divisions on the line ab produced either way; the graduations form the scale of the line, and its zero (which should evidently coincide with the point C) may be found by setting on ab produced three divisions of the scale (=the height of B above b) from b towards OX. Then the numbers on the scale will express the heights of the line AB above the plane XOY at the points of division to which they belong.

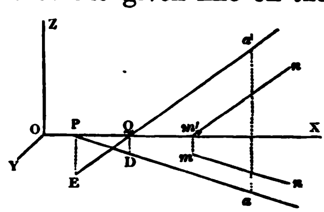
The point in which the line AB intersects the co-ordinate plane ZOX, supposed to be produced, if necessary, below OX, may be found thus: produce ab to meet OX in D, and draw

a line DF perpendicularly to OX to meet $a'c$ produced in F; the point F (conceived here to be in the plane ZOX produced below the plane XOY is the required intersection. This is manifest, because the projection on ZOX, of AB produced through C, will be the line $a'c$F; but the produced line ABC is in a plane perpendicular to YOX, and therefore must meet the plane ZOX produced in a line DF perpendicular to OX (DF being in the common section of the planes ZOX, and of that in which is the line ABC): consequently ABC produced must meet $a'c$ and DF at their point of section, that is at F.

The inclination of the line AB, in space, to ZOX may be found thus: from F or a' (suppose F) draw a line as FG at right angles to a'F, and make it equal to Ha, the perpendicular distance of A from the plane ZOX; then, if a', G be joined, the angle Ga'F will be the required inclination.

a. COR. To determine, on two co-ordinate planes, the projections of a line passing through a given point and parallel to a line which is given in space.

Let mn, $m'n'$ be the projections of the given line on the planes XOY and ZOX, and let a, a' be the projections of the given point on the same planes. Then, since (Art. 154. *b.*) when two lines in space are parallel to one another their projections on the co-ordinate planes are parallel to one another, through



a draw the line aP parallel to mn, meeting OX in P, and through a' draw a'E cutting OX in Q; these will be the required projections of the line. To find the points in which the line will intersect the co-ordinate planes, draw QD and PE each perpendicular to OX, the former meeting aP in D, and the latter meeting a'Q in E; the points D and E will, as explained in the proposition, be the intersections of the line with YOX and ZOX respectively, either of these planes being produced if necessary.

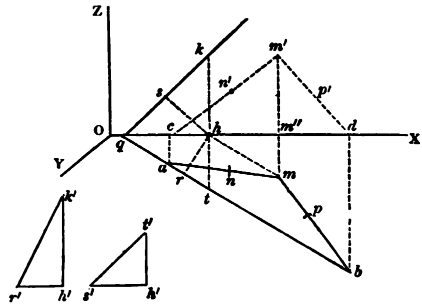
PROPOSITION II.

214. A plane in space, being conceived to pass through three given points, to determine its intersections with, and its inclinations to two co-ordinate planes.

Let the given points in space be designated M, N, P: let

their projections on XOY be m, n, p; and their projections on ZOX be m', n', p'.

Since M and N are in the required plane, a line joining M and N, and produced, will meet XOY in some point as a in the line mn produced; and since MNa is a straight line, its projection on ZOX will be a straight line passing through m', n'; therefore join m', n', and produce $m'n'$ till it meets OX in c. Draw ca perpendicular to OX, meeting mn produced in a; then a will be a point in the intersection of the required plane with XOY.



In like manner, join m', p', and produce $m'p'$ to meet OX in d; also draw db perpendicular to OX, meeting mp produced in b: then the line ba produced if necessary, will be that in which the required plane MNP intersects XOY.

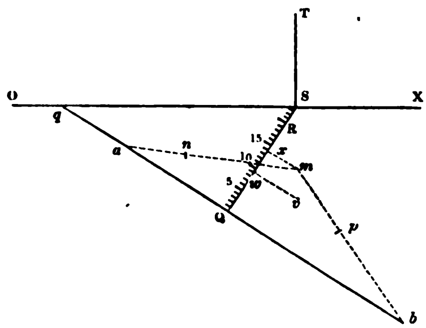
Draw mh parallel to ba: then mh will be the horizontal projection of a line parallel to ba, on the required plane, at a height equal to $m''m'$ vertically above XOY (mm'', $m''m'$ having been drawn perpendicularly to OX): draw hk perpendicular to OX, and make it equal to $m''m'$; then k will be a point in the intersection of the plane MNP with ZOX. Therefore, having produced ba till it meets OX in q, the line qk produced will be the intersection of MNP with ZOX.

The inclination of the plane bqk, that is MNP, to the plane XOY may be found thus. Draw hr perpendicular to bq; and on any convenient part of the paper take a line $r'h'$ equal to rh: draw next $h'k'$ perpendicular to $r'h'$ and equal to hk, and join r', k'; the angle $k'r'h'$ is evidently equal to the required inclination.

The inclination of bqk or MNP to ZOX may be found in a similar manner; thus, draw hs perpendicular to qk, and, on any convenient part of the paper take $h's'$ equal to hs; draw next $h't'$ perpendicular to $h's'$ and equal to ht (the latter having been drawn perpendicular to OX) and join s', t'. The angle $h's't'$ is evidently equal to the required inclination.

a. To form a scale of heights for a plane passing through three given points M, N, P, having the orthogonal projections m, n, p of those points on the plane of comparison XOY, and their heights above that plane.

Let, for example, the height of M above m be 14 feet, of N above n be 5 feet, and of P above p be 8 feet: join m, n, and m, p, and produce the lines indefinitely. Divide nm into 9 equal parts $(=14-5)$ and mp into six equal parts $(=14-8)$; and continue the graduations of mn and mp towards a and b, making na equal to five parts and pb equal to eight parts, according to the respective scales. Then a and b will be the zeros of the two scales of heights, and the line ab will be the intersection of the plane MNP with XOY.



Any straight lines joining the graduations distinguished by equal numbers, as 3 and 3, 5 and 5, &c. on the two scales, will be the orthogonal projections of horizontal lines on the plane MNP at the heights above the plane of comparison XOY expressed by those numbers.

Draw any line, as QR, perpendicular to ab, and a line mx from m parallel to ab, to meet QR: also divide Qx (Q being on ab) into 14 equal parts, the height of M above the plane of comparison. Let the points of division be numbered as in the figure; then QR will be a scale of heights for the plane MNP. In using the scale, let v be any point in the plane of comparison, and draw vw parallel to ab or perpendicular to QR meeting the latter in w, numbered 9; then a point in the plane MNP vertically over v or w will be 9 feet above the plane of comparison.

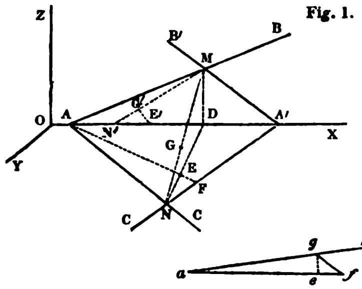
If ba be produced till it cut OX in q; then the intersection of MNP with ZOX will pass through q. Produce QR if necessary till it cut OX in S; the number at S (suppose 24) shows that the intersection of the plane MNP with the vertical co-ordinate plane ZOX passes through a point at the height of 24 feet vertically above S. Draw therefore ST perpendicular to OX and make ST equal to 24 feet, by the scale of horizontal distances; then a line drawn through q and T will represent the intersection of MNP with ZOX.

Proposition III.

215. To determine the angle at which two given planes in space are inclined to one another.

Let AB, AC be the intersections of one of the planes in space with the co-ordinate planes ZOX, YOX, and A′B′, A′C′ the intersections of the other plane with the same co-ordinate planes: then a line imagined to join the points of section M and N will represent the line in which the planes intersect one another.

Fig. 1.



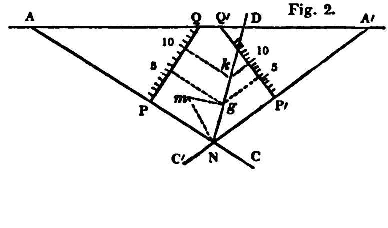
Draw MD perpendicular to OX and join N, D; also, from any point, as A in AC, draw AF perpendicular to ND cutting that line in E. Imagine a plane to pass through AF perpendicular to a line imagined to join M and N and cutting MN in some point as G: this plane being perpendicular to the two given planes, the angle AGF or its supplement will be their inclination to one another.

In order to determine this angle, first transfer the triangle MDN to the plane of the paper by making DN′ on OX equal to DN and joining N′, M; then make DE′ equal to DE, and draw E′G′ perpendicular to N′M. The point G′ will represent G. Again, transfer the triangle AGF to the plane of the paper, thus:—on a line *af* taken at pleasure make *ae*, *ef* equal to AE, EF respectively, and draw *eg* perpendicular to *af* making it equal to E′G′; complete the triangle *agf*, which will be the transferred triangle. Then the angle *agf* or its supplement *fgh* will be the required inclination of the planes BAC, B′A′C′ to one another.

The angle MN′D will, evidently, be equal to the inclination of the line MN to the plane XOY.

The line DN, or the projection of the line in which the two planes BAC, B′A′C′ intersect one another, may, by the scales of heights for those planes, be found thus. Let the intersections AC, A′C′ of the planes in space with the co-ordinate plane XOY be

Fig. 2.



found as in Prop. II. (Art. 214.), by means of three given points in each plane; and, by the known heights of two of the given points in each, form, in any convenient part of the paper, the scales of the two planes. Let the scales be PQ and P′Q′; these being, respectively, perpendicular to AC and A′C′, and P and P′ on those lines being the zero points: let the numbers be as in the figure.

Through any equal numbers, as 6 and 6 on the scales, draw lines parallel to AC and A′C′, or perpendicular to the scales, and intersecting one another in some point, as *g*: then *g* will be on the projection of the line MN, (Fig. 1.) in which the planes intersect one another; therefore join N, *g*, and produce N*g* to D in the line OX. The line ND may then be graduated, N being the zero point, and N*g* equal to six divisions; and thus is formed the scale of heights for the line MN of intersection.

It may be observed, that if the point N were not already obtained, it would be necessary to draw through other equal numbers, as 10 and 10 on PQ and P′Q′, lines perpendicular to the scales, and intersecting one another in some point as *k*; then a line drawn through *g* and *k* would be the projection of the line MN, Fig. 1.

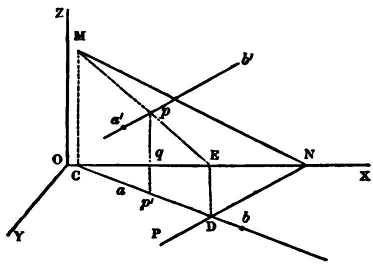
The inclination of MN to the plane XOY may be found by drawing at any point, as *g* in ND, Fig. 2., a line *gm* perpendicular to ND, and making it, by a scale of horizontal distances, equal to six feet (the height of MN vertically above *g*). Then, joining N, *m*, the angle DN*m* will be the required inclination.

a. Scholium. When two planes in space intersect a co-ordinate plane, as XOY, in two lines which are parallel to one another, the scales of the planes, which are perpendicular to those lines, will be also parallel to one another. If the graduations of the two scales are equal to one another, and the numbers increase in the like directions, the planes will be parallel to one another. If the scales are parallel to one another and the graduations are equal, but the numbers increase in opposite directions, the planes will be equally inclined to the co-ordinate plane, and they will intersect each other in a line parallel to that plane. The projection of the line of intersection is a line perpendicular to both scales, and passing through two points in them to which equal numbers are annexed.

Proposition IV.

216. To find a point in which a given line in space intersects a given plane.

Let MN, NP be the intersections of the given plane with the co-ordinate planes ZOX, XOY: let the given line be designated AB, and let *ab*, *a′b′* be its projections on XOY, ZOX. Produce *ba* till it meets OX in C, and draw CM perpendicular to OX; then CM will be the intersection with ZOX of a plane passing through AB perpendicular to XOY. At D, the intersection of NP with *ab*, draw DE perpendicular to OX and join M, E; then ME will be the intersection with ZOX of a plane passing through a line imagined to join M, D perpendicular to that co-ordinate plane.



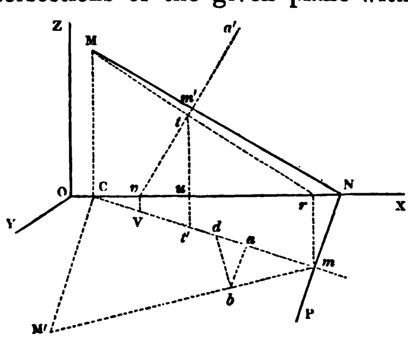
The required point of intersection must be somewhere in the line imagined to join M, D, for such line lies in the plane MNP, and the given line AB in space cuts it; therefore *p*, the intersection of ME with *a′b′*, is the point in which a line imagined to be drawn from the required point of intersection perpendicularly on ZOX would meet this plane; and drawing *pq*, *qp′* perpendicular to OX, *pq* will be the height of the point above YOX, while *p′* is the projection of the point on YOX.

Proposition V.

217. To find the point in which a line drawn through a given point perpendicularly to a given plane intersects that plane.

Let MN, NP be the intersections of the given plane with ZOX, YOX; and, the given point being designated A, let *a′*, *a* be the projections of the given point on the same planes.

Since the line is to be perpendicular to the plane MNP, its projections on YOX, ZOX will (Art. 159.) be perpendicular to NP, MN respectively; therefore,



N

through a and a' draw the lines am, $a'm'$ perpendicular to NP, MN; then these will be the projections of the perpendicular line on the co-ordinate planes. Produce ma till it meets OX in C, and draw CM perpendicular to OX, meeting MN in M; then the required point of intersection will be in a line imagined to join M, m. Draw mr perpendicular to OX, and join M, r; then the line Mr will be the projection of Mm on the plane ZOX. The point t, in which Mr intersects $a'm'$ produced, is the projection on ZOX of the point in which the perpendicular line cuts the given plane MNP; and drawing tu, ut' perpendicular to OX cutting Cm in t', the point t' will be the projection of the same point on XOY.

Produce $a'm'$ to cut OX in v, and draw vV cutting Cm in V; then V will be the point in which the line passing through the given point A perpendicularly to MNP will cut the plane XOY.

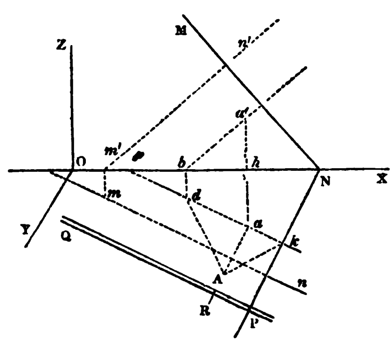
By the scale of heights. Through a draw maC perpendicular to NP; then, the inclination of the plane MNP to XOY being given, the scale of MNP is known: for example, the height of M above C is known (suppose it to be fifteen feet); then mC must be divided into fifteen equal parts for the scale of the plane, the zero being at m.

Now transfer the triangle MCm to the plane of the paper by drawing CM′ perpendicular to Cm and equal to fifteen feet (= CM) by the scale of horizontal distances, and joining M′, m. At a draw ab perpendicular to Cm meeting M′m in b, and draw bd perpendicular to M′m; then bd (supposed to be transferred to the plane MCm perpendicular to the paper) will be parallel to the line to be drawn from the given point A perpendicular to MNP. Graduate ad in equal parts as many as are expressed by the number at a on the scale of the plane (= the height of the plane vertically above a); then, since the graduations in the scales of heights for two parallel lines in space are equal to one another, the graduations of ad being continued towards C and numbered in a decreasing order, the number at a, expressing the known height of the given point A above a, will be the scale of the line perpendicular to MNP. The zero of these graduations will fall at some point (V) where the perpendicular cuts XOY; and the point (t'), where the numbers on the scale of MNP and the scale of the perpendicular line are the same, will be, on XOY, vertically under the point in which the perpendicular line from A cuts the plane MNP.

Proposition VI.

218. To determine the position of a plane which shall pass through a given point, and be perpendicular to a given line.

On the co-ordinate planes XOY, ZOY let a, a' be the projections of the given point, and mn, $m'n'$ projections of the given line.



Now if a line be supposed to pass through the given point parallel to the given line, it will be perpendicular to the required plane, and its projections on XOY, ZOX will be parallel to the projections of the given line on the same planes. Therefore, through a and a' draw lines parallel to mn and $m'n'$; and at b, where one of the former lines, as $a'b$, meets OX, draw bd perpendicular to OX, meeting the other in d: then d will be the point in which the line supposed to pass through the given point will cut XOY.

Draw aA perpendicular to ad, and make it equal to $a'h$ (the latter being perpendicular to OX); join d, A, and make Ak perpendicular to dA; then k will be a point in the intersection of the required plane with XOY. For the triangle dAk being supposed to turn on dk till it is perpendicular to the paper, A will be the given point, and Ad will be a line from A perpendicular to the required plane; therefore the plane of the triangle dAk will be perpendicular to the latter plane, Ak will be in the intersection of both planes, and the angle Akd will express the inclination of the required plane to XOY.

Through k draw PkN perpendicular to dk or mn, and from N in OX draw NM perpendicular to ba' or $m'n'$; then PN and MN will be the lines in which the required plane intersects XOY and ZOX. Thus the required plane is determined.

If the given line had passed through the given point, its projections in the directions ad and $a'b$, as well as the points a and a', would have been given, and the point k would have been determined by constructing the triangle dAk as above.

By a scale of heights. Let mn be the projection of the given line on the plane XOY, m being the point in which the

line intersects that plane; and, the height of any point in the given line above the plane XOY being given, let the line mn be graduated for a scale of the heights of that line, m being zero. Let a be the projection of the given point on XOY, and let the height of that point, and consequently of the required plane, vertically over a, be given (suppose 6 feet).

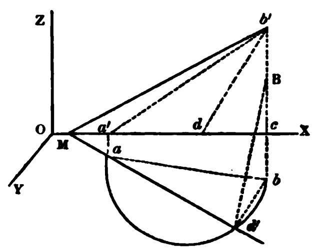
Draw dak parallel to mn through the point a, and make ad equal to six divisions of the scale on mn; then d will be the point in which a line passing through the given point parallel to the given line would meet XOY. Draw aA perpendicular to ak, and equal to 6 divisions (feet) of the scale of horizontal distances; join d, A, and draw Ak as before perpendicular to dA; then a line PN drawn through k perpendicular to dk will be the intersection of the required plane with XOY.

To make a scale for the plane. On any line PQ, drawn perpendicular to PN, make PR equal to ak (P being on PN), and divide it into six equal parts (the number of feet in the height of the plane above a); the graduations being continued beyond R, form the required scale, of which the zero is at the point P.

Proposition VII.

219. To determine the position of a plane which shall pass through a given line in space, and be inclined to one of the co-ordinate planes in a given angle.

Let ab, $a'b'$ be the projections of the given line on the co-ordinate planes XOY, ZOY, a being the point in which the given line cuts the former plane; so that aa' is perpendicular to OX. Join b, b; the line $b'cb$ will also be perpendicular to OX. Make the angle $cb'd$ equal to the complement of the given angle at which the required plane is to be inclined to the plane XOY;



and, on ab as a diameter, describe the semicircle $ad'b$; then, if the inscribed chord bd' be made equal to cd, the supplemental chord ad' will be in the intersection of the required plane with XOY. For, imagine a line, as bB, to be raised at b perpendicular to the plane XOY, and equal to $b'c$; then B will represent the point in space, of which b and b' are the projections: also, imagine B, d' to be joined, then the triangle Bbd' in space will be equal to $b'cd$, and the angle B$d'b$ to

$b'dc$, or the given inclination of the required plane to XOY. But ad' being perpendicular to $d'b$, is (Geom. Planes, 2. Def.) perpendicular to the plane $Bd'b$ in space, and therefore is the intersection with XOY of a plane passing through a, d' on the paper and B in space, and making with XOY an angle equal to that given inclination.

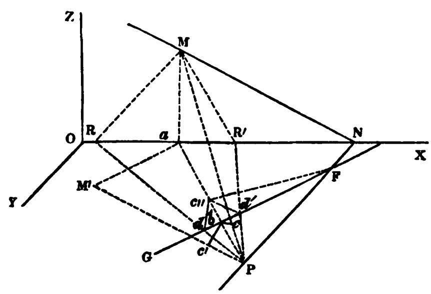
If $d'a$ be produced to meet OX, as in M, a line passing through M, b' will be the intersection of the required plane with the plane ZOX.

The scale of the plane $b'Md'$ may be formed on any line, as bd', perpendicular to ad'; the zero will be at d', and the line bd' must be divided into as many equal parts as are expressed by the number of feet or yards in $b'c$, that is, in the height of B above b.

Proposition VIII.

220. A plane being given, and a straight line situated in it; it is required to determine a plane which shall pass through that line, and make with the other plane a given angle.

Let MN, NP be the intersections of the given plane with the co-ordinate planes ZOX, YOX; and imagine MP, in space, to be the given line. Draw Ma perpendicular to OX, and join a, P; then aP will be the projection of MP on the plane YOX; also, draw any line FG perpendicular to aP, cutting the latter in b, and imagine a plane to pass through FG perpendicular to MP, cutting this line in some point c.



Transfer the triangle MaP to the plane YOX by drawing aM′ perpendicular to aP, making it equal to Ma, and joining M′, P.

Draw bc' perpendicular to M′P; then bc' will be equal to bc in the plane FcG. On Pa make bc'' equal to bc', and (F being the point in which FG cuts NP) join F, c''; then make the angle F$c''d$ or F$c''d'$ equal to the given inclination of the required plane to the plane MNP. Now the triangle F$c''d$, at present in the plane YOX, being conceived to turn

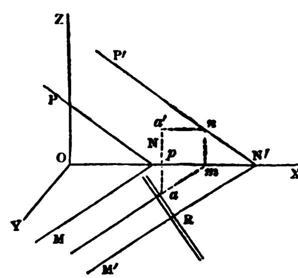
upon FG as an axis till it coincides with Fcd in the plane perpendicular to MP, it is evident that the required plane will pass through MP and through the point d or d'; therefore Pd produced to R, or Pd' produced to R′, the points R and R′ being in OX, will be the intersection of the required plane with YOX; and RM, or R′M, either of these lines being produced if necessary, will be its intersection with ZOX.

The like construction will evidently serve to determine a plane which shall be perpendicular to a given plane, and shall pass through a given line in the latter.

Proposition IX.

221. To determine a plane which shall pass through a given point, and be parallel to a given plane.

Let MN, NP be the intersections of the given plane with the co-ordinate planes XOY, ZOX, and let a, a' be the projections of the given point (designated A) on the same planes. Draw am parallel to MN, meeting OX in m, and from m draw mn perpendicular to OX. Make mn equal to $a'p$ ($a'p$ having been let fall from a' perpendicularly on OX). Through n draw P′N′ parallel to PN, and through N′ draw N′M′ parallel to MN; then the lines M′N′, N′P′ will be the intersections of the required plane with XOY, ZOX. It passes through A because a line imagined to be drawn through n parallel to M′N′ would lie in the plane, and would pass perpendicularly above a at a distance equal to $m\,n$ or $a'p$, either of which is equal to the height of A above a.

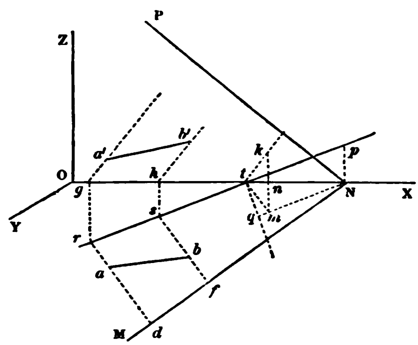


If the scale of the plane PNM were given or found, the line M′N′ might be found thus. The graduations of the scale for the required plane being, on account of the parallelism of the planes, equal to those of the given plane. On a line drawn through a perpendicular to MN, set from a a number of divisions equal to the number of feet in the height of the given point A above a (suppose 10), the divisions being taken from the scale of the given plane. Then, if R be the point at which the tenth division falls, the zero of the scale for M′N′P′ will be at R; therefore through R draw M′N′ for the line in which the required plane intersects XOY.

PROPOSITION X.

222. To determine a plane which shall pass through a given line, and be perpendicular to a given plane.

Let the given line be designated AB, and let ab, $a'b'$ be its projections on the co-ordinate planes YOX, ZOX; also, let MN, NP, be the intersections of the given plane with the same planes. Draw ad, $a'g$ perpendicular respectively to MN and NP; these will (Art. 159.) be the projections on the co-ordinate planes of a line passing through A perpendicularly to the plane MNP; and draw bf, $b'h$ perpendicular to MN, NP, for the projections of a line passing through B perpendicularly to MNP; then a plane supposed to pass through the perpendiculars falling from A and B will contain the given line, and be perpendicular to the given plane.



On da and fb produced find r, s, the points in which the perpendiculars from A and B will cut the plane YOX; then a line joining r and s will be the intersection of the required plane with that co-ordinate plane: let this line be produced to meet OX in t. Draw tk parallel to ga' or hb'; then tk will be the projection on ZOX of a line passing through t perpendicular to PNM; through any convenient point N in OX draw a line parallel to rt, and draw tm parallel to rd to meet it in m; draw mn perpendicular to OX, and produce it to meet tk in k; draw also Np perpendicular to OX, and make it equal to nk. Then a line drawn through t and p will be the intersection of the required plane with the co-ordinate plane ZOX. This is manifest; for a plane passing through rt, and a line through t, both perpendicular to PNM, will pass through a point perpendicularly above m at a distance equal to nk; also, since mN is parallel to rt, the plane will pass at an equal distance Np above N; therefore it will cut the plane ZOX in tp. Thus rt and tp are the intersections of the required plane with the co-ordinate planes.

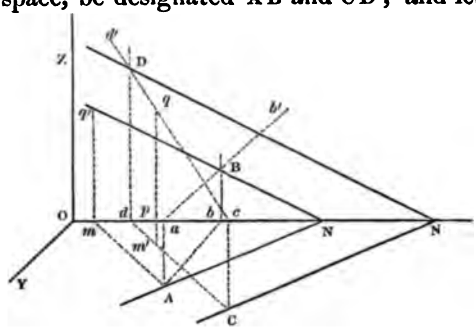
A scale for the required plane may be found by drawing tq perpendicular to rt, meeting Nm produced, if necessary,

in q, and dividing it into as many equal parts as are expressed by the number of feet, or yards, in the height kn, that height being measured by the scale of horizontal distances. The scale may be formed on any line perpendicular to rt, and the zero will be on the line rt.

Proposition XI.

223. To determine the positions of two planes which shall be parallel to one another, and each of which shall contain a given line; the lines not being in one plane.

Let the lines, in space, be designated AB and CD; and let Ab, ab' be the projections of AB; Cd cd' the projections of CD, A and C being the points in which the lines intersect the co-ordinate plane XOY.



Draw Am parallel to Cd, meeting OX in m; and on Cd, the greater of the two, take Cm' equal to Am, and draw $m'p$, pq perpendicular to OX, meeting cd' in q; draw mq' perpendicular to OX, and make it equal to pq; then q' will be the intersection with ZOX of a line, in space, parallel to CD, and passing through A.

Draw bB perpendicular to OX meeting ab' in B, and draw dD perpendicular to OX meeting cd' in D; the lines AB, CD in space will cut the plane ZOX in B and D respectively. Now a line drawn through q'B will be the intersection with ZOX of a plane passing through AB, and the line imagined to join A and q' which is parallel to CD: let it meet OX in N; then a line drawn from A to N will be the intersection of the same plane with XOY.

Draw CN′ parallel to AN, and join DN′; then CN′, DN′ will be the intersections with XOY, ZOX of a plane passing through CD in space parallel to the plane ANB. For N-$q'm$A, N′-DdC are two triangular pyramids, having their bases $q'm$A, DdC similar to one another (by the parallelism of the lines), also the surfaces NmA, N′dC similar to one another, and perpendicular to the bases; therefore (Geom. Prisms, &c., prop. 17.), the surfaces $q'm$N, dDN′ will be similar: consequently,

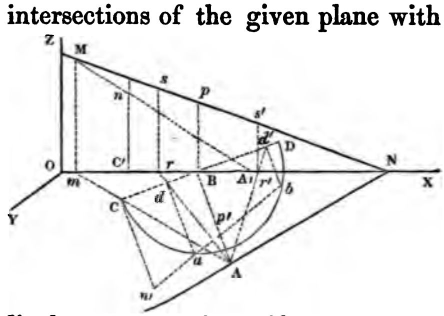
DN′ will be parallel to q′N. But (Art. 154. *a.*) two planes in space are parallel to one another when their intersections with each co-ordinate plane are parallel to one another; therefore ANB and CN′D are two parallel planes, of which one contains AB, and the other CD.

To form scales of heights for the two planes: graduate A*b* in as many equal parts as are expressed by the number of feet in the given height B*b*; and C*d* in as many equal parts as are expressed by the given height D*d*. These are the scales of heights for the lines AB, CD. Transfer the divisions of C*d* to A*m*, and number the divisions on A*b* and A*m* from the point A, which is zero; then lines joining points of division which have equal numbers on A*b*, A*m* (such lines will be parallel to AN), cutting a line drawn perpendicular to them, will form on this line a scale of heights for the plane ANB, the zero point being on AN. The same divisions will serve as a scale of heights for the plane CN′D, the zero point being on the line CN′, which is parallel to AN and to the lines joining the points which, on A*b* and A*m*, have equal numbers.

Proposition XII.

224. A plane in space being given, and from a point in its common section with one of the co-ordinate planes a line being drawn in the latter plane; it is required to determine, in the given plane, a line proceeding from the same point, and making with the former line a given angle.

Let AN, NM be the intersections of the given plane with the co-ordinate planes YOX, ZOX, and let AB be a given line in ZOX; it is required to determine a line in ANM which shall make with AB a given angle.



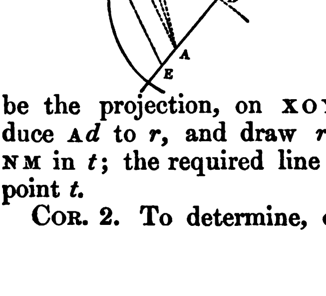
Make BAC equal to the given angle, and draw BC perpendicular to AB; then, if AC be supposed to revolve conically about AB, as an axis till it arrives in the plane ANM, it will in the latter position make with AB the same angle. Upon CB as a radius describe the semicircle C*a*D, which, though in the plane of the paper, must be conceived to be at right angles

to it, standing upon CD; produce AC to meet OX in *m*, and draw *m* M perpendicular to OX, meeting NM in M. Transfer the triangle M *m* A to the plane ZOX by making *m* A′ equal to *m* A and joining M, A′. Make A′ C′ equal to AC; and draw C′ *n* perpendicular to OX, meeting A′ M in *n*; then C′*n* will be the height of a point in the plane ANM vertically above C in the plane YOX. Draw C *n*′ perpendicular to BC and make it equal to C′*n*, also draw B*p* perpendicular to OX, and make B*p*′, on BA, equal to B*p*; then draw *n*′ *p*′ and produce it, and let it cut the semi-circumference in *a* and *b*. Next, draw *ad* perpendicular to BC; then a point vertically over *d* at a height equal to *da* will be the point in which C, by the revolution of AC, will meet the plane ANM; and A*d* will be the horizontal projection of the line on ANM, which is to make with AB the given angle. If A*d* be produced to cut OX in *r*, and *rs* be drawn perpendicular to OX, meeting MN in *s*, *s* will be the point in which the required line would meet NM, in the plane ZOX.

It is evident that the line AC, in revolving about AB would meet the semi-circumference in *b*; therefore, drawing *bd*′ perpendicular to CD and joining A, *d*′, A*d*′ will be the projection on XOY of a second line on the plane ANM, which would make with AB the given angle. Let A*d*′ meet OX in *r*′, and draw *r*′*s*′ perpendicular to OX; then this second line would meet the plane ZOX in *s*′.

COR. 1. If it were required to determine a line in the plane ANM, which, proceeding from a given point A in AN, shall make with that line a given angle, a corresponding process may be used; thus —

Make NAB equal to the given angle; let fall BB′ perpendicularly on AN, and on BB′ as a radius describe a segment of a circle; draw B*p*′ perpendicular to BB′, make it equal to B*p* (the latter having been drawn perpendicular to OX as in the figure to the proposition), and join *p*′B′. At *a*, where *p*′ B′ cuts the circumference, draw *ad* perpendicular to BB′; then the line A*d* will be the projection, on XOY, of the required line. Produce A*d* to *r*, and draw *rt* perpendicular to OX meeting NM in *t*; the required line will cut the plane ZOX in the point *t*.



COR. 2. To determine, on the plane ANM, a line pro-

ceeding from a given point A, which shall make with the co-ordinate plane XOY a given angle not exceeding the inclination of the planes to one another.

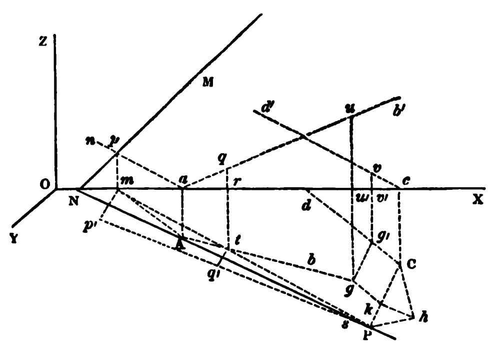
At any convenient point C in OX, make OCq equal to the given angle; and from q, in NM, draw qD perpendicular to OX; make DE equal to DC, cutting NA, produced if necessary, in E, and draw Ab parallel to DE. Then Ab will be the projection of the required line; and drawing bq' perpendicular to OX, a line imagined to join A and q' will manifestly make with the plane XOY an angle equal to OCq, that is, equal to the given angle.

Proposition XIII.

225. To determine a line which shall be perpendicular to two lines given, in space, and not lying in one plane.

Let the given lines in space be designated AB and CD; and, on the co-ordinate planes XOY, ZOX, let Ab, ab' be the projections of the first, Cd and cd' the projections of the other; A and C being the intersections of the two lines with the plane XOY.

Imagine a line, in space, to pass through A parallel to CD, and let its projections on the co-ordinate planes be Am, $a\,n$, these projections being (Art. 154. b.) parallel to Cd and $c\,d'$ respectively. A plane passing through this line, in space, will be parallel to CD; and it is required, next, to find the intersections of such plane with the co-ordinate planes, when the plane is made to pass through AB. For this purpose draw mp perpendicular to OX, meeting an, and take any point q in ab' such that if a perpendicular qr be let fall on OX, that perpendicular shall be less than pm; draw next qrt perpendicular to OX meeting Ab in t; and an indefinite line through m and t. Again, draw mp', tq' perpendicular to mt, making $mp' = mp$ and $tq' = rq$, and through p', q', a line meeting mt produced, in s. Then it is evident



that a plane in space, passing through AB and through the line which was imagined to pass through A parallel to CD will cut the plane XOY in a line drawn through A, *s*; such plane in space will be parallel to CD; also, if *s*A be produced to cut OX in N and a line NM be drawn through p (in which point the line parallel to CD cuts the plane ZOX) the same plane will cut ZOX in NM.

Now, if a perpendicular be conceived to be let fall on the plane ANM, from any point as C in the line CD, it will be equal to the perpendicular distance of that line from the line AB which is in the same plane; therefore, draw CP perpendicular to NA produced; and, the inclination of the plane ANM to XOY being found (Art. 214.) make the angle CPh equal to that inclination; then Ch drawn perpendicularly to Ph, and imagined to lie in a plane passing through CP perpendicularly to XOY, will be the distance from C to the plane ANM. Draw hk perpendicular to CP, then Ck will be the projection of Ch on the plane XOY. Draw kg parallel to Cd, meeting Ab in g, and gg' parallel to CP; then gg' will be the projection on XOY of a line conceived to be drawn from a point (of which point g' is the projection) in CD, perpendicularly to the plane ANM; consequently perpendicular to AB which it meets, in that plane, in a point of which g is the projection.

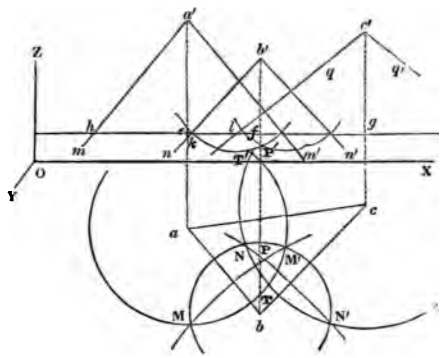
The projections, on ZOX, of the two points in AB and CD are evidently at u and v, where lines drawn from g and g' perpendicular to OX meet ab' and cd' respectively. The length of the required perpendicular line between AB and CD is manifestly equal to the hypotenuse of a right-angled triangle of which gg' and the difference between uu' and vv' are the two sides.

Proposition XIV.

226. Three points being given, in space, with their several angular distances, taken at a fourth point (not in the same plane as the others) from a vertical line passing through it; to determine the position of that fourth point.

Let the three given points be designated A, B, C, and let a, b, c be the projections of those points on XOY (supposed to be a horizontal plane); also, let a', b', c' be the projections of the same points on a vertical plane ZOX; then the lines $a'a$, $b'b$, $c'c$ will be perpendicular to OX.

Draw the indefinite lines $a'm$ and $a'm'$, $b'n$ and $b'n'$, $c'q$, and $c'q'$, the two first making with $a'a$, the two next with $b'b$ and the two last making with $c'c$ angles, each equal to the observed angular zenith distances of A, B, and C from the required point; these pairs of lines may be considered as comprehending the vertical sections of three cones of which A, B, C are the vertices, and whose curve surfaces will intersect one another in the required point.



Draw, next, any line, as hg, parallel to OX cutting the lines $a'm$, $b'm$, $c'q$ in points as h, k, l, and the lines $a'a$, $b'b$, $c'c$ in points as e, f, g; then eh, fk, gl may be considered as the radii of the sections of the three cones, made by a horizontal plane supposed to pass through hg. With a, b, c as centres, and radii equal to eh, fk, gl respectively, describe circles; these will represent the horizontal sections projected on XOY, and each pair of them will intersect one another in two points, as M and M$'$, N and N$'$, T and T$'$, in the figure. In like manner, drawing other lines parallel to OX, and from a, b, c as centres, with radii equal to the parts intercepted between $a'a$, $b'b$, $c'c$ and the sides of the cones, describing circles, each pair of these will intersect one another in two points corresponding to M and M$'$, N and N$'$, T and T$'$.

Then if three curve lines be drawn, one passing through all the points M, M$'$, another through all the points N and N$'$, and the third through all the points T and T$'$; these will represent the projections on XOY of the curve lines in which the three cones intersect one another; and a point as P in which all the three curves intersect one another will be, on the same plane, the projection of the required point.

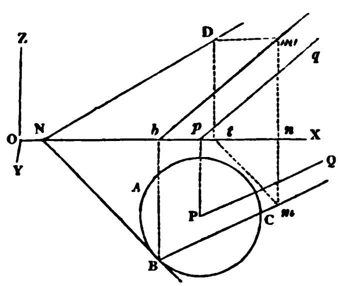
Again, perpendicularly to OX let lines be drawn from M and M$'$, N and N$'$, T and T$'$ to cut the line hg in points; and let corresponding lines be drawn from the intersections of the other pairs of circles till they cut the other lines corresponding to hg; then, on drawing three curve lines through the points, their common intersection P$'$ will be the projection on ZOX of the required point. In order to avoid confusion, only two of the three curve lines by which P and P$'$ are de-

termined, are shown in the figure. In strictness, two curves will suffice to determine P or P′; but, as the projections of curves of double curvature in which two cones intersect one another may have intersections in points which do not correspond to points on the surfaces of the cones, it will be a more certain and easy method to determine P or P′ by the common intersection of three projected curves.

Proposition XV.

227. To determine the position of a plane which, passing through a given point on the convex surface of a cylinder, may be a tangent to that surface.

Let ABC on the plane YOX be the circular base of a cylinder; and, imagining its axis to be oblique to the plane of the base, let PQ, pq be projections of the axis on YOX and ZOX. Let the given point on the surface of the cylinder be designated M; then a line imagined to be drawn through M parallel to the axis of the cylinder will coincide with a position of the line by whose revolution the cylinder is supposed to be described, and a plane touching the cylinder in M will touch it in the whole length of such line. The given projections of this line on XOY and ZOX will be parallel to PQ and pq respectively; and the first will meet the circle ABC in one point, or, being produced if necessary, will cut it in two points: let it cut the circle in two points as B and C; and let the line drawn through M parallel to the axis meet the base in the point B.



Now, since the required plane is to touch the cylinder in a line passing through M and B, its intersection with YOX will be a line touching the circle ABC in B; therefore draw BN so touching the circle, and let it cut OX in N. Next, if a line be conceived to be drawn on the tangent plane parallel to the plane XOY and passing through M, its projection on XOY (m being the projection of M on that plane) will be a line as mt parallel to BN, meeting OX in t; let mt be so drawn, and draw tD perpendicular to OX making it equal to nm'; (m' being the projection of M on ZOX) then the tangent plane will pass through D, and its intersection with ZOX will be

ND. Thus the intersections of the tangent plane with XOY and ZOX are determined.

If the line conceived to be drawn through M parallel to the axis of the cylinder had cut the base in C, a line touching ABC in C would have been the intersection of the tangent plane with XOY. The intersection of the tangent plane with ZOX might then be found in a manner similar to that by which ND was found.

If the cylinder had been upright, and M were any point on its convex surface, the projection of M on XOY would have been at some point on the circumference of the base, as B or C; then a tangent as BN would be the intersection of the tangent plane with XOY; and a line drawn through N perpendicular to OX would be the intersection on ZOX.

Proposition XVI.

228. To determine the position of a plane which, passing through a given point on the surface of a cone, may be a tangent to that surface.

Let ABC (Fig. to the preceding Proposition) be the circular base of a cone, supposed to be oblique, P its centre; and let PQ and pq be the projections of the axis on XOY and ZOX respectively, Q and q being the projections of the vertex. Then, the given point on the surface being designated M, a line supposed to be drawn from the vertex through M would meet the circumference of the base in some point as B or C. Lines drawn through Q and q, and through the projections of M on XOY, ZOX respectively, would be the projections of that line on those planes, and the former would determine the position of B or C.

The manner of determining the lines in which the required tangent plane would intersect YOX and ZOX, corresponds exactly to that which has been employed in the preceding proposition.

Proposition XVII.

229. To determine the position of a plane which, passing through a given point beyond a cylindrical or conical surface, may be a tangent to that surface.

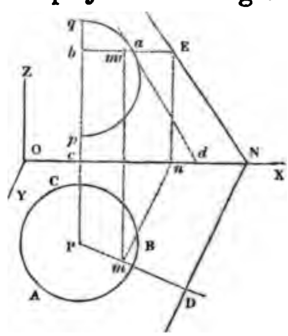
For a cylinder: imagine a straight line in space to pass through the given point parallel to the axis of the cylinder or to the generating line of the surface; then it is evident that the tangent plane must pass through this line. Find,

therefore (Art. 213. *a.*) the point in which a line drawn through the given point parallel to the axis of the cylinder will intersect the plane XOY, and from thence draw a line touching the base of the cylinder; this line will be the intersection of the required plane with XOY. Find next the intersection with ZOX of the line parallel to the axis; then a line joining that intersection with the point in which the tangent to the base meets OX, will be the intersection of the required plane with ZOX.

For a cone: find (Art. 213.) the points in which a line drawn through the given point and the vertex of the cone, will cut the planes XOY, ZOX; then a line drawn through the point in XOY, touching the base of the cone, will be the intersection of the required tangent plane with the same plane XOY, and a line drawn from the point in ZOX to that in which the tangent to the base meets OX will be the intersection of the required plane with ZOX.

Proposition XVIII.

230. To determine the position of a plane, which, passing through a given point on the surface of a sphere, may be a tangent to that surface.



Let ABC, on the plane XOY, be the projection of a great circle of the sphere parallel to that plane; let P be the centre of this circle, and, the line P*pq* being perpendicular to OX, let *pqa* be the projection on ZOX of half a great circle parallel to that plane.

Let the given point on the surface of the sphere be designated M, and let *m*, *m'* be its projections on XOY, ZOX respectively; also, imagine a section parallel to XOY to pass through M; its projection on XOY will be a circle whose circumference passes through *m*, and it has a line equal to P*m* for its semi-diameter, while *ba* passing through *m'* perpendicularly to *pq* will be the projection of a quadrant on ZOX.

Now a tangent to the section through M, parallel to XOY, will be a tangent to the sphere at that point; and its projection on XOY will be the line *mn* drawn perpendicular to P*m*. But the required tangent plane is to pass through M, it will therefore pass through the tangent, whose projection is *mn*,

and its intersection with XOY will be in a line parallel to *mn*. Draw *ad* touching the semicircle *qab* in *a*, and meeting OX in *d;* then if PD passing through *m* be made equal to *cd*, and DN be drawn parallel to *mn*, that line will be the intersection of the tangent plane with XOY. Next, since a tangent to the circular section passing through M, if conceived to be drawn from M till it meets ZOX, will be equal and parallel to *mn*, draw *n*E perpendicular to OX meeting *ba* produced in E; then the point E will be the intersection of the tangent line with ZOX, and a line drawn through N, E will be the intersection of the tangent plane with that co-ordinate plane.

The position of a plane touching any solid of revolution at a point given on that surface may be determined in like manner, if the axis of revolution be perpendicular to XOY, and a section passing through it parallel to ZOX be projected on the latter plane.

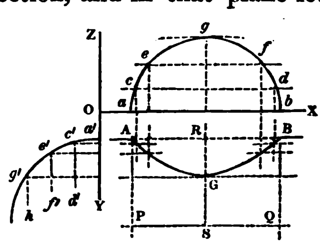
231. *Scholium.* The projections on XOY, ZOX of a normal, supposed to be drawn from a given point in a curve surface, are easily determined when the position of a plane touching the surface at that point has been found; since the projections are merely those of a line drawn from a given point in a plane perpendicularly to that plane. (See Art. 217.)

PROPOSITION XIX.

232. To determine the projection, on one of the co-ordinate planes, of the lines in which two cylinders may intersect one another.

Let the axes of the cylinders be at right angles to one another, and let them be in the plane on which the projection is to be made. For simplicity, let the bases of both cylinders be circles, but let their diameters be unequal.

Let XOY be the plane of projection, and in that plane let PQ, RS be the directions of the axes of the cylinders; also, let the semicircle *agb*, on ZOX, be the projection of the base of one of the hemicylinders, and *a'g'*, on ZOY, that of part of the base of the other hemicylinder. Through the smaller of the two semicircles draw any number of lines *cd*, *ef*, &c., parallel to OX; and through the greater, at

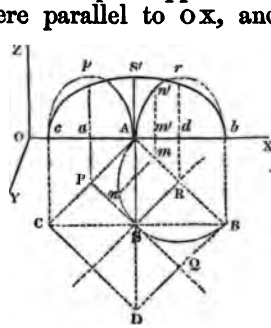


equal distances from OY, draw as many lines $c'd'$, $e'f'$, &c. perpendicular to OX. It is evident that these will be the projections, on ZOX and ZOY, of planes parallel to XOY; and if on AQ straight lines be drawn, as in the figure, through c, e, &c., c', e', &c. parallel both to OX and OY, these will be the projections on XOY of lines supposed to be drawn on the surfaces of the cylinders parallel to the two axes; the points in which they intersect one another respectively, being joined by a curve line, as in the figure, will be the required projection. A like curve must be understood to constitute the projection of the curve line in which the hemicylinders intersect one another on the side opposite to AB.

It is evident that those opposite curves will be similar to one another; it is also evident that if the diameters of the two circular cylinders were equal to one another, or if the bases of the cylinders were any similar and equal curves, the projection, on XOY, of the curves in which the cylinders intersect one another, would be two straight lines constituting the diagonals of the square or parallelogram formed on that plane by the chords of the vertical sections agb, &c.

If the axes of the two cylinders, though coinciding with the plane XOY, form oblique angles with one another and with the other co-ordinate planes, the problem would be solved in a similar manner by means of semicircles agb, $a'c'g'$ described on diameters perpendicular to the axes, and by drawing, parallel to those axes, the lines which determine points in the curves of intersection.

233. In the annexed figure, ABCD represents on XOY the projection of a groined vault, formed by the intersection of two hemicylinders having equal semicircular bases, and having their axes in the directions PQ and RS at right angles to one another. The figure within cS$'b$ is a projection, on ZOX, of half the vault; the curve cS$'b$ is the semi-ellipse supposed to stand vertically over CB, which is here parallel to OX, and the straight line AS$'$ is the projection of the elliptical quadrant over AS; the curves cpA, Arb are semi-ellipses which constitute the projections of the semicircles standing vertically over AC and AB.



Since the intersecting hemicylinders have equal diameters, each of the heights AS$'$, ap, dr, is equal to AP or AR, the radius of the semicircle standing on either

of those lines; Ab and Ac are each evidently equal to SB or SC.

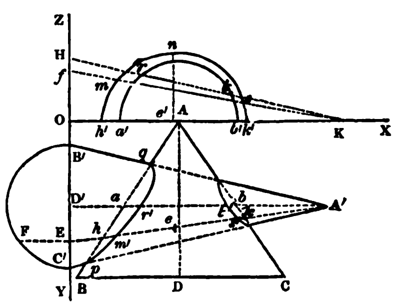
Whatever be the nature of the curves which, standing on AB and AC, form the bases of the intersecting hemicylinders, their projections on ZOX may be determined in the following manner: — Take any number of points, as m on AB, for example, and from each of them draw a line, as mn' perpendicular to OX; then mn being an ordinate to the given section AnSB of the hemicylinder, (that section being supposed to stand vertically over AB) make $m'n'$ equal to mn. The point n' will be in the required projection of AnSB. In like manner may any number of points in Arb, and also in Apc and cS$'b$, be found.

Proposition XX.

234. To determine the projections, on one of the co-ordinate planes, of the curve line in which two cones may intersect one another.

For simplicity let the cones have circular bases; let their axes be at right angles to one another, and let them be in the plane of projection.

Let ABC, A′B′C′ be the projections of the cones on the plane YOX, AD and A′D′ being the axes. Let $a'b'$ on OX be the projection of ab, in A′D′, and on $a'b'$ as a diameter describe a semicircle; this will be the projection on ZOX of a vertical section through the cone ABC on the diameter ab. On OZ make OH equal to D′B′ the semidiameter of the base of the cone; and on OX make OK equal to D′A′; then the triangle HOK will be the projection on ZOX of the half cone A′B′C′. Lines drawn perpendicular to OX from r and t, where HK cuts the semicircle on $a'b'$, will cut A′D′ in points r' and t', through which the required curves are to pass. The points p and q are evidently those in which the projection of one of the curves of intersection will meet AB.



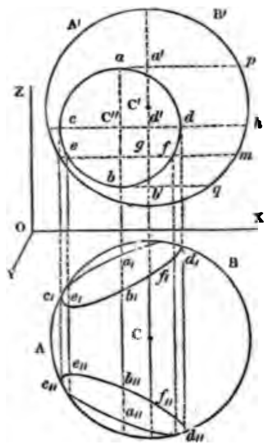
Take any point, as E in B′C′, and draw EA′, cutting AB and AC in h and k; also, draw the ordinate EF to a semicircle on B′C′ as a diameter. Make Of equal to EF, and

draw fK. On OX project hk and the point e in which it is bisected, in h', k', e', and describe a semi-ellipse on $h'k'$ as a transverse axis, with a semi-conjugate axis $e'n$ equal to the ordinate at e of a vertical section through the cone ABC, formed on a line drawn through e parallel to BC as a diameter; this curve will be intersected by fK in the points m and s, which, being projected on hk, will give two other points, m' and s', in the required curves. In like manner may any number of points be found.

Proposition XXI.

235. To determine the projections, on one of the co-ordinate planes, of the curve lines in which a cylinder may intersect a sphere.

The cylinder is supposed to have a circular base, and its axis to be parallel to the plane on which the projection is to be made.



Let AB be the projection of a great circle of the sphere on XOY, A′B′ the projection of a great circle on ZOX, C and C′ their centres, being in a line perpendicular to OX; also, let the circle $abcd$ be the projection of the base of the cylinder on the latter plane, C″ being its centre, and ab, cd diameters respectively perpendicular and parallel to OX.

Parallel to OX let the lines ap, ch, bq, and any others, as em, be drawn; these will represent the intersections on ZOX of the planes of circles of the sphere parallel to XOY. Let them intersect the line CC′, produced if necessary, in a', d', b', g, &c., and touch or cut the circle $acbd$ in a, b, c, d, f, &c.

Then, if lines perpendicular to OX be drawn through these last points, they will represent on YOX the projections of lines drawn on the surface of the cylinder parallel to its axis; and cutting the surface of the sphere on the circumferences of the small circles whose planes are projected in ap, ch, ef, &c. Therefore, with C as a centre, and a radius equal to $a'p$, describe arcs intersecting ab produced in $a_{,}$ and $a_{,,}$; also with C as a centre, and a radius equal to $b'q$, describe arcs intersecting ab produced in $b_{,}$ and $b_{,,}$. Next,

with C as a centre and a radius equal to $d'h$, describe arcs crossing the line drawn through c in $c_{,}$ and $c_{,,}$, and the line drawn through d in $d_{,}$ and $d_{,,}$; again, with C as a centre and a radius equal to gm, describe arcs crossing the line drawn through f in $f_{,}$ and $f_{,,}$; and so on. Curve lines $a_{,}\ c_{,}$ &c., $a_{,,}\ c_{,,}$ &c., drawn as in the figure, will be the required projections of the two curves in which the cylinder intersects the sphere.

The two curve lines are evidently similar to one another; they will touch the circumference of the circle AB in certain points near $c_{,}$ and $d_{,}$, $c_{,,}$ and $d_{,,}$; and it is evident that, if the eye of a spectator were above the plane YOX, the parts $c_{,}\ b_{,}\ d_{,}$ and $c_{,,}\ b_{,,}\ d_{,,}$ would be invisible.

If the axis of the cylinder pass through the centre of the sphere, the curves of intersection will evidently be circles, and their projections on XOY will be straight lines.

236. The following problems relate to the orthogonal projections, on vertical and horizontal planes, of the figures given to the voussoirs, or wrought stones which are employed in the construction of vaults of the most usual forms.

In the formation of voussoirs, it is of the utmost importance, when the mutual pressures of the stones are considerable, that the plane angles composing any solid angle should be either accurately, or as nearly as possible, right angles, or that the edges which are in the directions of the thickness of a vault should be in the positions of normals to the curve surface of the vault; for if the angles which the faces in contact, or joints, of any two voussoirs make with the faces which meet them are unequal, one of them, being less than a right angle, will be weaker than the other, and consequently, at the line of meeting, the stone is in danger of being splintered.

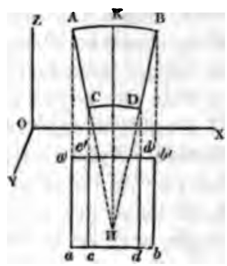
In simple vaults, such as the usual arches of a bridge, if the thickness be considered as equal throughout, the concave and the convex faces of each voussoir are parts of the cylindrical surfaces, circular or elliptical, of the vault; the contact faces or joints are planes, two of them perpendicular to the axes of the cylinder, and the two others normals to its surface, so that if the cylinder were circular, the planes of the last faces would pass through the axis. If a vault have the form of an upright cone, with a circular base, and be of equal thickness throughout, the concave and convex faces of each voussoir are parts of the conical surfaces; two of the joints are planes which, if produced, would pass through the axis of the vault, and each of the two others is a portion of the curve surface of a cone the circumference of whose base is the exterior edge of the horizontal course of voussoirs of

which the one considered is a part, and whose vertex is in the axis, all lines imagined to be drawn on these conical surfaces from the base to the vertex being normals to both surfaces of the vault. Again, if the vault be a spherical dome everywhere of equal thickness, the concave and convex faces of each voussoir will be portions of the spherical surfaces, while the joints will be similar to the corresponding joints of a conical vault, except that those which have the conical form will have the centre of the dome for their common vertex.

Proposition XXII.

237. To describe the projections, on a vertical and a horizontal plane, of the key-voussoir in a hemicylindrical vault, the vertical plane of projection being perpendicular to the axis of the cylinder, which, for simplicity, is supposed to be circular.

Let HK on ZOX be a vertical line meeting the axis of the vault in H; then H will be the centre of the circular arcs CD and AB, on the concave and convex faces of the voussoir. The radii HC, HA are given; therefore the arcs can be described. The chord of half the arc CD or AB being given, when set on both sides of HK, it will determine the points through which the radii HA, HB are to be drawn for the sides of the voussoir. The lines aa', cc', &c. drawn, as in the figure, through A, C, &c. perpendicularly to OX, if cut by the lines ab, $a'b'$ parallel to OX at distances from one another equal to the given depth of the voussoir, will determine the projection of that voussoir on the horizontal plane XOY.

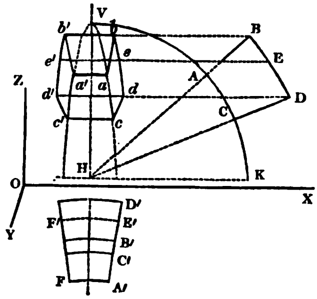


If the cylindrical vault were elliptical, and if the arcs CD and AB had the same centre of curvature, the point H would have been the intersection of the produced sides AC and BD, each of which would have been in the direction of a normal to either of those arcs.

PROPOSITION XXIII.

238. To describe the projections, on a vertical and a horizontal plane, of a voussoir forming part of a hemispherical dome.

Let ABCD be a vertical section through such a voussoir in a plane ZOX, on which the vertical projection is to be made; let HV be the axis, and H the centre of the dome. Through A, C, B, D draw lines, as in the figure, perpendicular to HV; these lines will represent the intersections on ZOX of the planes of the circles between which the horizontal course of voussoirs, of which ABCD is a section, is contained.



From any convenient point as a centre in the line VH produced, and with radii equal to the perpendicular distances of A, C, B, D from HV, draw the arcs at A′, C′, &c.; these will be the projections on XOY of parts of the circles just mentioned. Then a straight line A′F, being made equal to the given chord of the voussoir on the circumference of the circle passing through A′ (half the chord being set on each side of VH produced), and lines being drawn from the centre through A′ and F, there will be determined the figure FD′, which is the projection of the voussoir on the plane XOY.

Now if, parallel to VH, lines be drawn through A′, C′, B′, D′ and the corresponding points on the opposite side of VH produced till they meet the corresponding straight lines drawn through A, C, B, D as above; they will determine the points a, c, b, d and a', c', b', d' in the projection on ZOX; but the normals BA, DC, &c. tending to the centre of the dome, it is evident that, in the projection on ZOX, ba, $b'a'$, dc, $d'c'$ will be straight lines all tending to H. The circular arcs AC, BD, &c. will, in the projection on ZOX be evidently portions of ellipses, since they are orthogonal projections of parts of great circles of the sphere on a plane inclined to their own planes; but any number of points in those lines may be determined in the following manner: — Take a point as E in BD, and draw Ee' perpendicular to VH; with a radius equal to the perpendicular distance of E, from VH describe an arc E′F′ concentric with A′F; then half the chord of that arc

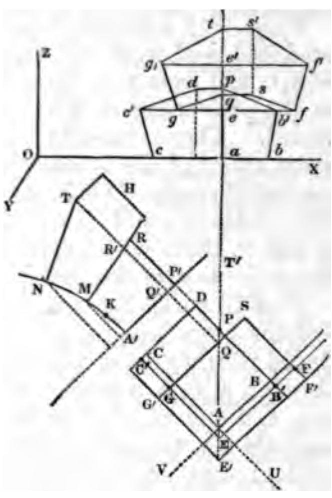
being set on E e', on each side of VH, the points e, e' will be in the elliptical curve passing through b and d, b' and d'. The figure $abdc$ $a'b'd'c'$ is the required projection of the voussoir on the plane ZOX.

The projection of a voussoir for a conical vault would differ from that which has been described only by the lines ba and dc, $b'a'$ and $d'c'$, which are perpendicular to the surface of the cone, being respectively parallel to one another, and by the lines corresponding to ac, bd, $a'c'$, $b'd'$, both in the vault and in the projection being straight.

PROPOSITION XXIV.

239. To describe the projections, on a vertical and a horizontal plane, of a voussoir at an angle of a groined vault, or one formed by the intersection of two hemicylindrical vaults.

Let AV, AU be part of two sides of a square formed on a horizontal plane XOY when two hemicylindrical vaults of equal magnitudes intersect one another at right angles; XOY being the plane in which are the axes of the cylinders. For simplicity let the cylinders be circular, and let A′N, supposed to be in a plane perpendicular to the paper and cutting it in A′P′ parallel to AV, represent part of the arch line in a semicircular section taken vertically over AV; also, let MR, NT indicate the positions of two voussoir joints which are normals to the hemicylinder of which A′N is a section.



Let T′A produced be the direction of a diagonal of the square on AV, AU; and let the produced part represent the direction, on the plane XOY, of the curve line (ellipse in the present case) in which the hemicylinders intersect one another. Let CABD, C′EB′D be horizontal projections of the lower and upper surfaces of the voussoir which rests immediately on XOY, GE being in a line drawn through M parallel to AU, and EB′ being parallel to AV. A section

through this voussoir in a vertical plane passing through a line parallel to AV is represented by MA′P′R. Again, let GEFS, G′E′F′S be a horizontal projection of the voussoir next above the other, G′E′ being in a line drawn through N parallel to AU, and E′F′ being parallel to AV. A section through this voussoir in a vertical plane passing through AV is represented by the figure NMH.

Let the vertical plane ZOX, on which the groined voussoirs are to be represented, be perpendicular to the line AT′ in YOX; and on AT′, produced to cut OX, let the point a be the representation of A. On at, in the direction of AT′ produced, make ae, ae' equal, respectively, to the perpendicular distances of M and N from P′A′ produced; and through e and e' draw indefinite lines parallel to OX; these will be the projections on ZOX of the horizontal lines passing through points corresponding to M and N parallel to AB and AC on the intersecting concave faces of the vault.

Making ab and ac, also eb' and ec' respectively equal to the perpendicular distances of B and C, B′ and C′ from AT′; the points b and c, b' and c' will be representations of B and C. B′ and C′ on the lower and upper joints of the voussoir A′MRP′, The curves bb', cc' are representations of the lines in which the vertical joints of that voussoir meet the faces of the vaults; these being representations of portions of circles projected on a plane, which is oblique to their planes, are portions of ellipses; and any number of points in them may be found by means of points, as K, taken in A′M, as b' and c' were found by means of the point M.

To determine the projection, on ZOX, of the upper face of the voussoir MA′P′R; on at make ap equal to P′R; then the straight line $b'p$ will be the projection of the edge vertically over B′P; draw pd parallel to OX, and make it equal to the perpendicular distance of D from AT′; this will be the projection of the (horizontal) edge vertically over PD; and the straight line $c'd$ will be the projection of the edge vertically above C′D.

In the upper surface of this voussoir, the plane, of which $b'ep$ is the projection, is one which if produced would pass through the axis of the hemicylinder vertically over the space UAB, that axis being parallel to VAB; and the plane of which $epdc'$ is the projection, is one which if produced would pass through the axis of the hemicylinder vertically over VAC, that axis being parallel to UAC.

The figure $gqsf$ is the projection, on ZOX, of the inferior surface of the voussoir, of which NMH is a section; it is determined by first making ef and eg respectively equal to the

perpendicular distance of F and G from AT′; next, by making aq equal to Q′R′, in the line QT parallel to AC; then drawing qs parallel to OX, and making it equal to the perpendicular distance of S from AT. The line gq (parallel to $c'd$) is the projection of the edge vertically above GQ, qs is the projection of the (horizontal) edge above QS, and sf (parallel to pb') is the projection of the edge above SF; the plane of which gqe is the projection coincides with that of which $c'pde$ is the projection, and the plane of $eqsf$ with that of epb'.

The projection of the upper surface of the voussoir NMH is determined in a similar manner; thus, on the indefinite line drawn through e', parallel to OX, make $e'f'$, $e'g'$ respectively equal to the perpendicular distances of F′ and G′ from AT′; make at equal to Q′T, then draw ts' parallel to OX, and make it equal to qs, or the perpendicular distance of S from AT′. The line $g't$ is the projection of the edge vertically above G′Q, ts' that of the (horizontal) edge above QS, and $s'f'$ that of the edge above SF′; also, the plane of which $g'te'$ is the projection is one which if produced would pass through the axis of the hemicylinder over the space VAC, and that of which $e'ts'f'$ is the projection is one which if produced would pass through the axis of the cylinder over the space UAB. The curve lines ff', gg', like bb', cc', are portions of ellipses.

THE END.

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